CHAPTER 19
Basic Numerical Procedures

Notes for the Instructor

Chapter 19 presents the standard numerical procedures used to value derivatives when analytic results are not available. These involve binomial/trinomial trees, Monte Carlo simulation, and finite difference methods.

Binomial trees are introduced in Chapter 11, and Section 19.1 and 19.2 can be regarded as a review and more in-depth treatment of that material. When covering Section 19.1, I usually go through in some detail the calculations for a number of nodes in an example such as the one in Figure 19.3. Once the basic tree building and roll back procedure has been covered it is fairly easy to explain how it can be extended to currencies, indices, futures, and stocks that pay dividends. Also the calculation of hedge statistics such as delta, gamma, and vega can be explained. The software DerivaGem is a convenient way of displaying trees in class as well as an important calculation tool for students.

The binomial tree and Monte Carlo simulation approaches use risk-neutral valuation arguments. By contrast, the finite difference method solves the underlying differential equation directly. However, as explained in the book the explicit finite difference method is essentially the same as the trinomial tree method and the implicit finite difference method is essentially the same as a multinomial tree approach where there are $M + 1$ branches emanating from each node. Binomial trees and finite difference methods are most appropriate for American options; Monte Carlo simulation is most appropriate for path-dependent options.

Any of Problems 19.25 to 19.30 work well as assignment questions.

QUESTIONS AND PROBLEMS

Problem 19.1.

Which of the following can be estimated for an American option by constructing a single binomial tree: delta, gamma, vega, theta, rho?

Delta, gamma, and theta can be determined from a single binomial tree. Vega is determined by making a small change to the volatility and recomputing the option price using a new tree. Rho is calculated by making a small change to the interest rate and recomputing the option price using a new tree.

Problem 19.2.

Calculate the price of a three-month American put option on a non-dividend-paying stock when the stock price is $60, the strike price is $60, the risk-free interest rate is 10%
per annum, and the volatility is 45% per annum. Use a binomial tree with a time interval of one month.

In this case, \( S_0 = 60, K = 60, r = 0.1, \sigma = 0.45, T = 0.25, \) and \( \Delta t = 0.0833. \) Also
\[
\begin{align*}
    u &= e^{\sigma \sqrt{\Delta t}} = e^{0.45\sqrt{0.0833}} = 1.1387 \\
    d &= \frac{1}{u} = 0.8782 \\
    a &= e^{r\Delta t} = e^{0.1 \times 0.0833} = 1.0084 \\
    p &= \frac{a - d}{u - d} = 0.4998 \\
    1 - p &= 0.5002
\end{align*}
\]

The output from DerivaGem for this example is shown in the Figure S19.1. The calculated price of the option is $5.16.

Figure S19.1  Tree for Problem 19.2

Problem 19.3.

Explain how the control variate technique is implemented when a tree is used to value American options.
The control variate technique is implemented by

(a) valuing an American option using a binomial tree in the usual way (\( = f_A \)).
(b) valuing the European option with the same parameters as the American option using
the same tree (\( = f_E \)).
(c) valuing the European option using Black–Scholes (\( = f_{BS} \)). The price of the American
option is estimated as \( f_A + f_{BS} - f_E \).

**Problem 19.4.**

*Calculate the price of a nine-month American call option on corn futures when the current futures price is 198 cents, the strike price is 200 cents, the risk-free interest rate is 8% per annum, and the volatility is 30% per annum. Use a binomial tree with a time interval of three months.*

In this case \( F_0 = 198 \), \( K = 200 \), \( r = 0.08 \), \( \sigma = 0.3 \), \( T = 0.75 \), and \( \Delta t = 0.25 \). Also

\[
\begin{align*}
  u &= e^{0.3 \sqrt{0.25}} = 1.1618 \\
  d &= \frac{1}{u} = 0.8607 \\
  a &= 1 \\
  p &= \frac{a - d}{u - d} = 0.4626 \\
  1 - p &= 0.5373
\end{align*}
\]

The output from DerivaGem for this example is shown in the Figure S19.2. The calculated price of the option is 20.34 cents.

**Problem 19.5.**

*Consider an option that pays off the amount by which the final stock price exceeds
the average stock price achieved during the life of the option. Can this be valued using
the binomial tree approach? Explain your answer.*

A binomial tree cannot be used in the way described in this chapter. This is an example of what is known as a history-dependent option. The payoff depends on the path followed by the stock price as well as its final value. The option cannot be valued by starting at the end of the tree and working backward since the payoff at the final branches is not known unambiguously. Chapter 26 describes an extension of the binomial tree approach that can be used to handle options where the payoff depends on the average value of the stock price.

**Problem 19.6.**

*"For a dividend-paying stock, the tree for the stock price does not recombine; but the
tree for the stock price less the present value of future dividends does recombine." Explain
this statement.*
Figure S19.2  Tree for Problem 19.4

Suppose a dividend equal to $D$ is paid during a certain time interval. If $S$ is the stock price at the beginning of the time interval, it will be either $Su - D$ or $Sd - D$ at the end of the time interval. At the end of the next time interval, it will be one of $(Su - D)u$, $(Su - D)d$, $(Sd - D)u$ and $(Sd - D)d$. Since $(Su - D)d$ does not equal $(Sd - D)u$ the tree does not recombine. If $S$ is equal to the stock price less the present value of future dividends, this problem is avoided.

**Problem 19.7.**

Show that the probabilities in a Cox, Ross, and Rubinstein binomial tree are negative when the condition in footnote 9 holds.

With the usual notation

\[
p = \frac{a - d}{u - d} \\
1 - p = \frac{u - a}{u - d}
\]

If $a < d$ or $a > u$, one of the two probabilities is negative. This happens when

\[
e^{(r-q)\Delta t} < e^{-\sigma \sqrt{\Delta t}}
\]
or
\[ e^{(r-q)\Delta t} > e^{\sigma \sqrt{\Delta t}} \]
This in turn happens when \((q-r)\sqrt{\Delta t} > \sigma\) or \((r-q)\sqrt{\Delta t} > \sigma\) Hence negative probabilities occur when
\[ \sigma < |(r-q)\sqrt{\Delta t}| \]
This is the condition in footnote 9.

**Problem 19.8.**

*Use stratified sampling with 100 trials to improve the estimate of \(\pi\) in Business Snapshot 19.1 and Table 19.1.*

In Table 19.1 cells A1, A2, A3, ..., A100 are random numbers between 0 and 1 defining how far to the right in the square the dart lands. Cells B1, B2, B3, ..., B100 are random numbers between 0 and 1 defining how high up in the square the dart lands. For stratified sampling we could choose equally spaced values for the A's and the B's and consider every possible combination. To generate 100 samples we need ten equally spaced values for the A's and the B's so that there are \(10 \times 10 = 100\) combinations. The equally spaced values should be 0.05, 0.15, 0.25, ..., 0.95. We could therefore set the A's and B's as follows:

\[
A1 = A2 = A3 = \ldots = A10 = 0.05
\]
\[
A11 = A12 = A13 = \ldots = A20 = 0.15
\]
\[
\ldots
\]
\[
A91 = A92 = A93 = \ldots = A100 = 0.95
\]

and

\[
B1 = B11 = B21 = \ldots = B91 = 0.05
\]
\[
B2 = B12 = B22 = \ldots = B92 = 0.15
\]
\[
\ldots
\]
\[
B10 = B20 = B30 = \ldots = B100 = 0.95
\]

We get a value for \(\pi\) equal to 3.2, which is closer to the true value than the value of 3.04 obtained with random sampling in Table 19.1. Because samples are not random we cannot easily calculate a standard error of the estimate.

**Problem 19.9.**

*Explain why the Monte Carlo simulation approach cannot easily be used for American-style derivatives.*
In Monte Carlo simulation sample values for the derivative security in a risk-neutral world are obtained by simulating paths for the underlying variables. On each simulation run, values for the underlying variables are first determined at time $\Delta t$, then at time $2\Delta t$, then at time $3\Delta t$, etc. At time $i\Delta t$ ($i = 0, 1, 2 \ldots$) it is not possible to determine whether early exercise is optimal since the range of paths which might occur after time $i\Delta t$ have not been investigated. In short, Monte Carlo simulation works by moving forward from time $t$ to time $T$. Other numerical procedures which accommodate early exercise work by moving backwards from time $T$ to time $t$.

**Problem 19.10.**

A nine-month American put option on a non-dividend-paying stock has a strike price of $49. The stock price is $50, the risk-free rate is 5% per annum, and the volatility is 30% per annum. Use a three-step binomial tree to calculate the option price.

In this case, $S_0 = 50$, $K = 49$, $r = 0.05$, $\sigma = 0.30$, $T = 0.75$, and $\Delta t = 0.25$. Also

$$u = e^{\sigma \sqrt{\Delta t}} = e^{0.30 \sqrt{0.25}} = 1.0126$$
$$d = \frac{1}{u} = 0.8607$$
$$a = e^{r \Delta t} = e^{0.05 \times 0.0833} = 1.0084$$
$$p = \frac{a - d}{u - d} = 0.5043$$
$$1 - p = 0.4957$$

The output from DerivaGem for this example is shown in the Figure S19.3. The calculated price of the option is $4.29. Using 100 steps the price obtained is $3.91

**Problem 19.11.**

Use a three-time-step tree to value a nine-month American call option on wheat futures. The current futures price is 400 cents, the strike price is 420 cents, the risk-free rate is 6%, and the volatility is 35% per annum. Estimate the delta of the option from your tree.

In this case $F_0 = 400$, $K = 420$, $r = 0.06$, $\sigma = 0.35$, $T = 0.75$, and $\Delta t = 0.25$. Also

$$u = e^{0.35 \sqrt{0.25}} = 1.1912$$
$$d = \frac{1}{u} = 0.8395$$
$$a = 1$$
$$p = \frac{a - d}{u - d} = 0.4564$$
$$1 - p = 0.5436$$

The output from DerivaGem for this example is shown in the Figure S19.4. The calculated price of the option is 42.07 cents. Using 100 time steps the price obtained is 38.64. The

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Growth factor per step, $a = 1.0126$
Probability of up move, $p = 0.5043$
Up step size, $u = 1.1618$
Down step size, $d = 0.8607$

Figure S19.3  Tree for Problem 19.10

options delta is calculated from the tree is

$$(79.971 - 11.419)/(476.498 - 335.783) = 0.487$$

When 100 steps are used the estimate of the option's delta is 0.483.

Problem 19.12.

A three-month American call option on a stock has a strike price of $20. The stock price is $20, the risk-free rate is 3% per annum, and the volatility is 25% per annum. A dividend of $2 is expected in 1.5 months. Use a three-step binomial tree to calculate the option price.

In this case the present value of the dividend is $2e^{-0.03 \times 0.125} = 1.9925$. We first build a tree for $S_0 = 20 - 1.9925 = 18.0075$, $K = 20$, $r = 0.03$, $\sigma = 0.25$, and $T = 0.25$ with $\Delta t = 0.08333$. This gives Figure S19.5. For nodes between times 0 and 1.5 months we then add the present value of the dividend to the stock price. The result is the tree in

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Growth factor per step, \( a = 1.0000 \)
Probability of up move, \( p = 0.4564 \)
Up step size, \( u = 1.1912 \)
Down step size, \( d = 0.8395 \)

\[
\begin{array}{c}
\text{Node Time:} \\
0.0000 & 0.2500 & 0.5000 & 0.7500 \\
\end{array}
\]

**Figure S19.4** Tree for Problem 19.11

Figure S19.6. The price of the option calculated from the tree is 0.674. When 100 steps are used the price obtained is 0.690.

**Problem 19.13.**

A one-year American put option on a non-dividend-paying stock has an exercise price of $18. The current stock price is $20, the risk-free interest rate is 15% per annum, and the volatility of the stock price is 40% per annum. Use the DerivaGem software with four 3-month time steps to estimate the value of the option. Display the tree and verify that the option prices at the final and penultimate nodes are correct. Use DerivaGem to value the European version of the option. Use the control variate technique to improve your estimate of the price of the American option.

In this case \( S_0 = 20, K = 18, r = 0.15, \sigma = 0.40, T = 1, \) and \( \Delta t = 0.25. \) The
Time step, $dt = 0.0833$ years, 30.42 days
Growth factor per step, $a = 1.0025$
Probability of up move, $p = 0.4993$
Up step size, $u = 1.0748$
Down step size, $d = 0.9304$

![Figure S19.5 First tree for Problem 19.12](image)

Node Time:

0.0000  0.0833  0.1667  0.2500

The tree produced by DerivaGem for the American option is shown in Figure S19.7. The estimated value of the American option is $1.29.

As shown in Figure S19.8, the same tree can be used to value a European put option with the same parameters. The estimated value of the European option is $1.14. The option parameters are $S = 20$, $K = 18$, $r = 0.15$, $\sigma = 0.40$ and $T = 1$

$$d_1 = \frac{\ln(20/18) + 0.15 + 0.40^2/2}{0.40} = 0.8384$$
$$d_2 = d_1 - 0.40 = 0.4384$$
$$N(-d_1) = 0.2009; \quad N(-d_2) = 0.3306$$

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Time step, $dt = 0.0833$ years, 30.42 days
Growth factor per step, $a = 1.0025$
Probability of up move, $p = 0.4993$
Up step size, $u = 1.0748$
Down step size, $d = 0.9304$

![Figure S19.6 Final Tree for Problem 19.12](image)

Node Time:
- 0.0000
- 0.0833
- 0.1667
- 0.2500

The true European put price is therefore

$$18e^{-0.15} \times 0.3306 - 20 \times 0.2009 = 1.10$$

The control variate estimate of the American put price is therefore $1.29 + 1.10 - 1.14 = 1.25$.

**Problem 19.14.**

A two-month American put option on a stock index has an exercise price of 480. The current level of the index is 484, the risk-free interest rate is 10% per annum, the dividend yield on the index is 3% per annum, and the volatility of the index is 25% per annum. Divide the life of the option into four half-month periods and use the tree approach to estimate the value of the option.

In this case $S_0 = 484$, $K = 480$, $r = 0.10$, $\sigma = 0.25$, $q = 0.03$, $T = 0.1667$, and
Growth factor per step, $a = 1.0382$
Probability of up move, $p = 0.5451$
Up step size, $u = 1.2214$
Down step size, $d = 0.8187$

Node Time:
0.0000 0.2500 0.5000 0.7500 1.0000

Figure S19.7  Tree to evaluate American option for Problem 19.13

Growth factor per step, $a = 1.0382$
Probability of up move, $p = 0.5451$
Up step size, $u = 1.2214$
Down step size, $d = 0.8187$

Node Time:
0.0000 0.2500 0.5000 0.7500 1.0000

Figure S19.8  Tree to evaluate European option in Problem 19.13
\[ \Delta t = 0.04167 \]

\[ u = e^{\sigma \sqrt{\Delta t}} = e^{0.25 \sqrt{0.04167}} = 1.0524 \]

\[ d = \frac{1}{u} = 0.9502 \]

\[ a = e^{(r-q)\Delta t} = 1.00292 \]

\[ p = \frac{a - d}{u - d} = \frac{1.0029 - 0.9502}{1.0524 - 0.9502} = 0.516 \]

The tree produced by DerivaGem is shown in the Figure S19.9. The estimated price of the option is $14.93.

**Problem 19.15.**

*How can the control variate approach improve the estimate of the delta of an American option when the tree approach is used?*

First the delta of the American option is estimated in the usual way from the tree. Denote this by \( \Delta_A^* \). Then the delta of a European option which has the same parameters as the American option is calculated in the same way using the same tree. Denote this by \( \Delta_B^* \). Finally the true European delta, \( \Delta_B \), is calculated using the formulas in Chapter 17. The control variate estimate of delta is then:

\[ \Delta_A^* - \Delta_B^* + \Delta_B \]
Problem 19.16.

Suppose that Monte Carlo simulation is being used to evaluate a European call option on a non-dividend-paying stock when the volatility is stochastic. How could the control variate and antithetic variable technique be used to improve numerical efficiency? Explain why it is necessary to calculate six values of the option in each simulation trial when both the control variate and the antithetic variable technique are used.

In this case a simulation requires two sets of samples from standardized normal distributions. The first is to generate the volatility movements. The second is to generate the stock price movements once the volatility movements are known. The control variate technique involves carrying out a second simulation on the assumption that the volatility is constant. The same random number stream is used to generate stock price movements as in the first simulation. An improved estimate of the option price is

\[ f_A^* - f_B^* + f_B \]

where \( f_A^* \) is the option value from the first simulation (when the volatility is stochastic), \( f_B^* \) is the option value from the second simulation (when the volatility is constant) and \( f_B \) is the true Black-Scholes value when the volatility is constant.

To use the antithetic variable technique, two sets of samples from standardized normal distributions must be used for each of volatility and stock price. Denote the volatility samples by \( \{V_1\} \) and \( \{V_2\} \) and the stock price samples by \( \{S_1\} \) and \( \{S_2\} \). \( \{V_1\} \) is antithetic to \( \{V_2\} \) and \( \{S_1\} \) is antithetic to \( \{S_2\} \). Thus if

\( \{V_1\} = +0.83, +0.41, -0.21 \ldots \)

then

\( \{V_2\} = -0.83, -0.41, +0.21 \ldots \)

Similarly for \( \{S_1\} \) and \( \{S_2\} \).

An efficient way of proceeding is to carry out six simulations in parallel:

- Simulation 1: Use \( \{S_1\} \) with volatility constant
- Simulation 2: Use \( \{S_2\} \) with volatility constant
- Simulation 3: Use \( \{S_1\} \) and \( \{V_1\} \)
- Simulation 4: Use \( \{S_1\} \) and \( \{V_2\} \)
- Simulation 5: Use \( \{S_2\} \) and \( \{V_1\} \)
- Simulation 6: Use \( \{S_2\} \) and \( \{V_2\} \)

If \( f_i \) is the option price from simulation \( i \), simulations 3 and 4 provide an estimate \( 0.5(f_3 + f_4) \) for the option price. When the control variate technique is used we combine this estimate with the result of simulation 1 to obtain \( 0.5(f_3 + f_4) - f_1 + f_B \) as an estimate of the price where \( f_B \) is, as above, the Black-Scholes option price. Similarly simulations 2, 5 and 6 provide an estimate \( 0.5(f_5 + f_6) - f_2 + f_B \). Overall the best estimate is:

\[ 0.5[0.5(f_3 + f_4) - f_1 + f_B + 0.5(f_5 + f_6) - f_2 + f_B] \]
Problem 19.17.

Explain how equations (19.27) to (19.30) change when the implicit finite difference method is being used to evaluate an American call option on a currency.

For an American call option on a currency

$$\frac{\partial f}{\partial t} + (r - r_f)S \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf$$

With the notation in the text this becomes

$$\frac{f_{i+1,j} - f_{ij}}{\Delta t} + (r - r_f)j \Delta S \frac{f_{i,j+1} - f_{i,j-1}}{2 \Delta S} + \frac{1}{2} \sigma^2 j^2 \Delta S^2 \frac{f_{i,j+1} - 2f_{i,j} + f_{i,j-1}}{\Delta S^2} = rf_{ij}$$

for $j = 1, 2 \ldots M - 1$ and $i = 0, 1 \ldots N - 1$. Rearranging terms we obtain

$$a_j f_{i,j-1} + b_j f_{ij} + c_j f_{i,j+1} = f_{i+1,j}$$

where

$$a_j = \frac{1}{2} (r - r_f) j \Delta t - \frac{1}{2} \sigma^2 j^2 \Delta t$$

$$b_j = 1 + \sigma^2 j^2 \Delta t + r \Delta t$$

$$c_j = -\frac{1}{2} (r - r_f) j \Delta t - \frac{1}{2} \sigma^2 j^2 \Delta t$$

Equations (19.28), (19.29) and (19.30) become

$$f_{N,j} = \max [j \Delta S - K, 0] \quad j = 0, 1 \ldots M$$

$$f_{i0} = 0 \quad i = 0, 1 \ldots N$$

$$f_{IM} = M \Delta S - K \quad i = 0, 1 \ldots N$$

Problem 19.18.

An American put option on a non-dividend-paying stock has four months to maturity. The exercise price is $21, the stock price is $20, the risk-free rate of interest is 10% per annum, and the volatility is 30% per annum. Use the explicit version of the finite difference approach to value the option. Use stock price intervals of $4 and time intervals of one month.

We consider stock prices of $0, $4, $8, $12, $16, $20, $24, $28, $32, $36 and $40. Using equation (19.34) with $r = 0.10$, $\Delta t = 0.0833$, $\Delta S = 4$, $\sigma = 0.30$, $K = 21$, $T = 0.3333$ we obtain the table shown below. The option price is $1.56.

<table>
<thead>
<tr>
<th>Stock Price ($)</th>
<th>Time To Maturity (Months)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>4</td>
</tr>
<tr>
<td>40</td>
<td>0.00</td>
</tr>
<tr>
<td>36</td>
<td>0.00</td>
</tr>
<tr>
<td>32</td>
<td>0.01</td>
</tr>
<tr>
<td>28</td>
<td>0.07</td>
</tr>
<tr>
<td>24</td>
<td>0.38</td>
</tr>
<tr>
<td>20</td>
<td>1.56</td>
</tr>
<tr>
<td>16</td>
<td>5.00</td>
</tr>
<tr>
<td>12</td>
<td>9.00</td>
</tr>
<tr>
<td>8</td>
<td>13.00</td>
</tr>
<tr>
<td>4</td>
<td>17.00</td>
</tr>
<tr>
<td>0</td>
<td>21.00</td>
</tr>
</tbody>
</table>

Problem 19.19.

The spot price of copper is $0.60 per pound. Suppose that the futures prices (dollars per pound) are as follows:

- 3 months: 0.59
- 6 months: 0.57
- 9 months: 0.54
- 12 months: 0.50

The volatility of the price of copper is 40% per annum and the risk-free rate is 6% per annum. Use a binomial tree to value an American call option on copper with an exercise price of $0.60 and a time to maturity of one year. Divide the life of the option into four 3-month periods for the purposes of constructing the tree. (Hint: As explained in Section 14.7, the futures price of a variable is its expected future price in a risk-neutral world.)

In this case $\Delta t = 0.25$ and $\sigma = 0.4$ so that

$$u = e^{0.4\sqrt{0.25}} = 1.2214$$

$$d = \frac{1}{u} = 0.8187$$

The futures prices provide estimates of the growth rate in copper in a risk-neutral world. During the first three months this growth rate (with continuous compounding) is

$$4 \ln \frac{0.59}{0.60} = -6.72\% \text{ per annum}$$

The parameter $p$ for the first three months is therefore

$$\frac{e^{-0.0672\times0.25} - 0.8187}{1.2214 - 0.8187} = 0.4088$$
Figure S19.10  Tree to value option in Problem 19.19: At each node, upper number is price of copper and lower number is option price.

The growth rate in copper is equal to $-13.79\%$, $-21.63\%$ and $-30.78\%$ in the following three quarters. Therefore, the parameter $p$ for the second three months is

$$e^{-0.1379 \times 0.25} - 0.8187 \over 1.2214 - 0.8187 = 0.3660$$

For the third quarter it is

$$e^{-0.2163 \times 0.25} - 0.8187 \over 1.2214 - 0.8187 = 0.3195$$

For the final quarter, it is

$$e^{-0.3078 \times 0.25} - 0.8187 \over 1.2214 - 0.8187 = 0.2663$$

The tree for the movements in copper prices in a risk-neutral world is shown in Figure S19.10. The value of the option is $0.062$. 

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Problem 19.20.

Use the binomial tree in Problem 19.19 to value a security that pays off $x^2$ in one year where $x$ is the price of copper.

In this problem we use exactly the same tree for copper prices as in Problem 19.19. However, the values of the derivative are different. On the final nodes the values of the derivative equal the square of the price of copper. On other nodes they are calculated in the usual way. The current value of the security is $0.275$ (see Figure S19.11).

Problem 19.21.

When do the boundary conditions for $S = 0$ and $S \rightarrow \infty$ affect the estimates of derivative prices in the explicit finite difference method?

Define $S_t$ as the current asset price, $S_{\text{max}}$ as the highest asset price considered and $S_{\text{min}}$ as the lowest asset price considered. (In the example in the text $S_{\text{min}} = 0$). Let

$$Q_1 = \frac{S_{\text{max}} - S_t}{\Delta S} \quad \text{and} \quad Q_2 = \frac{S_t - S_{\text{min}}}{\Delta S}$$
and let $N$ be the number of time intervals considered. From the structure of the calculations in the explicit version of the finite difference method, we can see that the values assumed for the derivative security at $S = S_{\text{min}}$ and $S = S_{\text{max}}$ affect the derivative security’s value at time $t$ if

$$N \geq \max(Q_1, Q_2)$$

**Problem 19.22.**

*How would you use the antithetic variable method to improve the estimate of the European option in Business Snapshot 19.2 and Table 19.2?*

The following changes could be made. Set LI as

$$= \text{NORMSINV}(	ext{RAND}())$$

A1 as

$$= \text{C}$\	ext{EXP}((\text{E}$\	ext{2} - \text{F}$\	ext{2}$\	ext{2})*\text{G}$\	ext{2} + \text{F}$\	ext{2}$\	ext{1}*$\text{L}$\	ext{2}*$\text{SQRT}($\text{G}$\	ext{2}))$$

H1 as

$$= \text{C}$\	ext{EXP}((\text{E}$\	ext{2} - \text{F}$\	ext{2}$\	ext{2})*\text{G}$\	ext{2} - \text{F}$\	ext{2}$\	ext{1}*$\text{L}$\	ext{2}*$\text{SQRT}($\text{G}$\	ext{2}))$$

I1 as

$$= \text{EXP}(-\text{E}$\	ext{2}$\	ext{G}$\	ext{2})*\text{MAX}(\text{H}$\	ext{1} - \text{D}$\	ext{2}, 0)$$

and J1 as

$$= 0.5*(\text{B}$\	ext{1} + \text{J}$\	ext{1})$$

Other entries in columns L, A, H, and I are defined similarly. The estimate of the value of the option is the average of the values in the J column.

**Problem 19.23.**

*A company has issued a three-year convertible bond that has a face value of $25 and can be exchanged for two of the company’s shares at any time. The company can call the issue when the share price is greater than or equal to $18. Assuming that the company will force conversion at the earliest opportunity, what are the boundary conditions for the price of the convertible? Describe how you would use finite difference methods to value the convertible assuming constant interest rates. Assume there is no risk of the company defaulting.*

The basic approach is similar to that described in Section 19.8. The only difference is the boundary conditions. For a sufficiently small value of the stock price, $S_{\text{min}}$, it can be assumed that conversion will never take place and the convertible can be valued as a straight bond. The highest stock price which needs to be considered, $S_{\text{max}}$, is $18$. When this is reached the value of the convertible bond is $36$. At maturity the convertible is worth the greater of $2S_T$ and $25$ where $S_T$ is the stock price.

The convertible can be valued by working backwards through the grid using either the explicit or the implicit finite difference method in conjunction with the boundary conditions. In formulas (19.25) and (19.32) the present value of the income on the convertible between time $t + i \Delta t$ and $t + (i + 1) \Delta t$ discounted to time $t + i \Delta t$ must be added to the right-hand side. Chapter 26 considers the pricing of convertibles in more detail.

Provide formulas that can be used for obtaining three random samples from standard normal distributions when the correlation between sample \( i \) and sample \( j \) is \( \rho_{i,j} \).

Suppose \( x_1 \), \( x_2 \), and \( x_3 \) are random samples from three independent normal distributions. Random samples with the required correlation structure are \( \epsilon_1, \epsilon_2, \epsilon_3 \) where

\[
\epsilon_1 = x_1
\]

\[
\epsilon_2 = \rho_{12} x_1 + x_2 \sqrt{1 - \rho_{12}^2}
\]

and

\[
\epsilon_3 = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3
\]

where

\[
\alpha_1 = \rho_{13}
\]

\[
\alpha_1 \rho_{12} + \alpha_2 \sqrt{1 - \rho_{12}^2} = \rho_{23}
\]

and

\[
\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1
\]

This means that

\[
\alpha_1 = \rho_{13}
\]

\[
\alpha_2 = \frac{\rho_{23} - \rho_{13} \rho_{12}}{\sqrt{1 - \rho_{12}^2}}
\]

\[
\alpha_3 = \sqrt{1 - \alpha_1^2 - \alpha_2^2}
\]

ASSIGNMENT QUESTIONS

Problem 19.25.

An American put option to sell a Swiss franc for dollars has a strike price of $0.80 and a time to maturity of one year. The volatility of the Swiss franc is 10%, the dollar interest rate is 6%, the Swiss franc interest rate is 3%, and the current exchange rate is 0.81. Use a three-time-step tree to value the option. Estimate the delta of the option from your tree.

The binomial tree is shown in Figure M19.1. The value of the option is estimated as 0.0207, and its delta is estimated as

\[
\frac{0.006221 - 0.041153}{0.858142 - 0.764559} = -0.3733
\]

A one-year American call option on silver futures has an exercise price of $9.00. The current futures price is $8.50, the risk-free rate of interest is 12% per annum, and the volatility of the futures price is 25% per annum. Use the DerivaGem software with four three-month time steps to estimate the value of the option. Display the tree and verify that the option prices at the final and penultimate nodes are correct. Use DerivaGem to value the European version of the option. Use the control variate technique to improve your estimate of the price of the American option.

In this case \( F_0 = 8.5, K = 9, r = 0.12, T = 1, \sigma = 0.25, \) and \( \Delta t = 0.25. \) The parameters for the tree are

\[
\begin{align*}
  u &= e^{\sigma \sqrt{\Delta t}} = e^{0.25 \sqrt{0.25}} = 1.1331 \\
  d &= \frac{1}{u} = 0.8825 \\
  a &= 1 \\
  p &= \frac{a - d}{u - d} = \frac{1 - 0.8825}{1.1331 - 0.8825} = 0.469
\end{align*}
\]

The tree output by DerivaGem for the American option is shown in Figure M19.2. The estimated value of the option is $0.596. The tree produced by DerivaGem for the European version of the option is shown in Figure M19.3. The estimated value of the option is $0.586. The Black-Scholes price of the option is $0.570. The control variate estimate of the price of the option is therefore 

\[
0.596 + 0.570 - 0.586 = 0.580
\]
At each node:
Upper value = Underlying Asset Price
Lower value = Option Price
Shaded values are a result of early exercise.

Strike price = 0.8
Discount factor per step = 0.9802
Time step, \( dt = 0.3333 \) years, 121.67 days
Growth factor per step, \( a = 1.0101 \)
Probability of up move, \( p = 0.5726 \)
Up step size, \( u = 1.0594 \)
Down step size, \( d = 0.9439 \)

Node Time:
0.0000  0.3333  0.6667  1.0000

Figure M19.1  Tree for Problem 19.25
At each node:
Upper value = Underlying Asset Price
Lower value = Option Price
Shaded values are a result of early exercise.

Strike price = 9
Discount factor per step = 0.9704
Time step, \( dt = 0.2500 \text{ years, } 91.25 \text{ days} \)
Growth factor per step, \( a = 1.0000 \)
Probability of up move, \( p = 0.4688 \)
Up step size, \( u = 1.1331 \)
Down step size, \( d = 0.8825 \)

Node Time:
0.0000 0.2500 0.5000 0.7500 1.0000

Figure M19.2  Tree for American option in Problem 19.26

Problem 19.27.
A six-month American call option on a stock is expected to pay dividends of $1 per share at the end of the second month and the fifth month. The current stock price is $30, the exercise price is $34, the risk-free interest rate is 10% per annum, and the volatility of the part of the stock price that will not be used to pay the dividends is 30% per annum. Use the DerivaGem software with the life of the option divided into six time steps to estimate the value of the option. Compare your answer with that given by Black's approximation (see Section 13.12).

DerivaGem gives the value of the option as 0.989. Black's approximation sets the price of the American call option equal to the maximum of two European options. The first lasts the full six months. The second expires just before the final ex-dividend date. In this case the software shows that the first European option is worth 0.957 and the second is worth 0.997. Black's model therefore estimates the value of the American option as 0.997. This is close to the tree value of 0.989.
At each node:
Upper value = Underlying Asset Price
Lower value = Option Price
Shaded values are a result of early exercise.

Strike price = 9
Discount factor per step = 0.9704
Time step, $dt = 0.2500$ years, 91.25 days
Growth factor per step, $a = 1.0000$
Probability of up move, $p = 0.4688$
Up step size, $u = 1.1331$
Down step size, $d = 0.8825$

Figure M19.3  Tree for European option in Problem 19.26

Problem 19.28.

The current value of the British pound is $1.60 and the volatility of the pound-dollar exchange rate is 15% per annum. An American call option has an exercise price of $1.62 and a time to maturity of one year. The risk-free rates of interest in the United States and the United Kingdom are 6% per annum and 9% per annum, respectively. Use the explicit finite difference method to value the option. Consider exchange rates at intervals of 0.20 between 0.80 and 2.40 and time intervals of 3 months.

In this case equation (19.34) becomes

$$f_{ij} = a_j^* f_{i+1,j-1} + b_j^* f_{i+1,j} + c_j^* f_{i+1,j+1}$$
where

\[ a_j^* = \frac{1}{1 + r\Delta t} \left[ \frac{1}{2} (r - r_f) j \Delta t + \frac{1}{2} \sigma^2 j^2 \Delta t \right] \]

\[ b_j^* = \frac{1}{1 + r\Delta t} (1 - \sigma^2 j^2 \Delta t) \]

\[ c_j^* = \frac{1}{1 + r\Delta t} \left[ \frac{1}{2} (r - r_f) j \Delta t + \frac{1}{2} \sigma^2 j^2 \Delta t \right] \]

The parameters are \( r = 0.06, r_f = 0.09, \sigma = 0.15, S = 1.60, K = 1.62, T = 1, \Delta t = 0.25, \Delta S = 0.2 \) and we obtain the table shown below. The option price is \$0.062.

<table>
<thead>
<tr>
<th>Stock Price</th>
<th>12</th>
<th>9</th>
<th>6</th>
<th>3</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.40</td>
<td>0.780</td>
<td>0.780</td>
<td>0.780</td>
<td>0.780</td>
<td>0.780</td>
</tr>
<tr>
<td>2.20</td>
<td>0.580</td>
<td>0.580</td>
<td>0.580</td>
<td>0.580</td>
<td>0.580</td>
</tr>
<tr>
<td>2.00</td>
<td>0.380</td>
<td>0.380</td>
<td>0.380</td>
<td>0.380</td>
<td>0.380</td>
</tr>
<tr>
<td>1.80</td>
<td>0.180</td>
<td>0.180</td>
<td>0.180</td>
<td>0.180</td>
<td>0.180</td>
</tr>
<tr>
<td>1.60</td>
<td>0.062</td>
<td>0.054</td>
<td>0.043</td>
<td>0.027</td>
<td>0.000</td>
</tr>
<tr>
<td>1.40</td>
<td>0.011</td>
<td>0.007</td>
<td>0.003</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>1.20</td>
<td>0.001</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>1.00</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>0.80</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
</tbody>
</table>

Problem 19.29.

Answer the following questions concerned with the alternative procedures for constructing trees in Section 19.4.

a. Show that the binomial model in Section 19.4 is exactly consistent with the mean and variance of the change in the logarithm of the stock price in time \( \Delta t \).

b. Show that the trinomial model in Section 19.4 is consistent with the mean and variance of the change in the logarithm of the stock price in time \( \Delta t \) when terms of order \((\Delta t)^2\) and higher are ignored.

c. Construct an alternative to the trinomial model in Section 19.4 so that the probabilities are 1/6, 2/3, and 1/6 on the upper, middle, and lower branches emanating from each node. Assume that the branching is from \( S \) to \( Su \), \( Sm \), or \( Sd \) with \( m^2 = ud \). Match the mean and variance of the change in the logarithm of the stock price exactly.

(a) For the binomial model in Section 19.4 there are two equally likely changes in the logarithm of the stock price in a time step of length \( \Delta t \). These are \((r - \sigma^2/2)\Delta t + \sigma \sqrt{\Delta t}\) and \((r - \sigma^2/2)\Delta t - \sigma \sqrt{\Delta t}\). The expected change in the logarithm of the stock price is

\[ 0.5[(r - \sigma^2/2)\Delta t + \sigma \sqrt{\Delta t}] + 0.5[(r - \sigma^2/2)\Delta t - \sigma \sqrt{\Delta t}] = (r - \sigma^2/2)\Delta t \]
This is correct. The variance of the change in the logarithm of the stock price is
\[ 0.5\sigma^2 \Delta t + 0.5\sigma^2 \Delta t = \sigma^2 \Delta t \]
This is correct.
(b) For the trinomial tree model in Section 19.4, the change in the logarithm of the stock price in a time step of length \( \Delta t \) is \( \pm \sigma \sqrt{3\Delta t} \), 0, and \( -\sigma \sqrt{3\Delta t} \) with probabilities
\[
\sqrt{\frac{\Delta t}{12\sigma^2}} \left( r - \frac{\sigma^2}{2} \right) + \frac{1}{6}, \quad \frac{2}{3}, \quad -\sqrt{\frac{\Delta t}{12\sigma^2}} \left( r - \frac{\sigma^2}{2} \right) + \frac{1}{6}
\]
The expected change is
\[ \left( r - \frac{\sigma^2}{2} \right) \Delta t \]
Its variance is \( \sigma^2 \Delta t \) plus a term of order \((\Delta t)^2\). These are correct.
(c) To get the expected change in the logarithm of the stock price in time \( \Delta t \) correct we require
\[
\frac{1}{6}(\ln u) + \frac{2}{3}(\ln m) + \frac{1}{6}(\ln d) = \left( r - \frac{\sigma^2}{2} \right) \Delta t
\]
The relationship \( m^2 = ud \) implies \( \ln m = 0.5(\ln u + \ln d) \) so that the requirement becomes
\[ \ln m = \left( r - \frac{\sigma^2}{2} \right) \Delta t \]
or
\[ m = e^{(r - \sigma^2)\Delta t} \]
The expected change in \( \ln S \) is \( \ln m \). To get the variance of the change in the logarithm of the stock price in time \( \Delta t \) correct we require
\[
\frac{1}{6}(\ln u - \ln m)^2 + \frac{1}{6}(\ln d - \ln m)^2 = \sigma^2 \Delta t
\]
Because \( \ln u - \ln m = -(\ln d - \ln m) \) it follows that
\[ \ln u = \ln m + \sigma \sqrt{3\Delta t} \]
\[ \ln d = \ln m - \sigma \sqrt{3\Delta t} \]
These results imply that
\[
m = e^{(r - \sigma^2)\Delta t}
\]
\[ u = e^{(r - \sigma^2)\Delta t + \sigma \sqrt{3\Delta t}} \]
\[ d = e^{(r - \sigma^2)\Delta t - \sigma \sqrt{3\Delta t}} \]
Problem 19.30.

The DerivaGem Application Builder functions enable you to investigate how the prices of options calculated from a binomial tree converge to the correct value as the number of time steps increases. (See Figure 19.4 and Sample Application A in DerivaGem.) Consider a put option on a stock index where the index level is 900, the strike price is 900, the risk-free rate is 5%, the dividend yield is 2%, and the time to maturity is 2 years.

a. Produce results similar to Sample Application A on convergence for the situation where the option is European and the volatility of the index is 20%.

b. Produce results similar to Sample Application A on convergence for the situation where the option is American and the volatility of the index is 20%.

c. Produce a chart showing the pricing of the American option when the volatility is 20% as a function of the number of time steps when the control variate technique is used.

d. Suppose that the price of the American option in the market is 85.0. Produce a chart showing the implied volatility estimate as a function of the number of time steps.

See the charts following.
Problem 19.30b

Problem 19.30c
Notes for the Instructor

Some instructors may prefer to cover Chapter 21 before Chapter 20 because the estimation of volatilities and correlations is necessary for the model building approach for calculating VaR. This works well. I prefer to do Chapter 20 first because VaR has become such a fundamental measure. After doing Chapter 20 students understand why Chapter 21 is important.

When the model building approach is covered it should be emphasized that we are relating the actual change in the value of the portfolio to percentage changes in the values of the market variables. The use of the model building approach can be presented in the context of the classic work of Markowitz on portfolio selection. I like to spend some time on the simple examples in Section 20.3. This leads on to the linear model in Section 20.4. Tables 20.3 and 20.4 form a starting point for discussing principal components analysis and provide the data for a number of end-of-chapter problems. Principal components analysis is also used in the discussion of interest rate models in Chapter 31.

Any of Problems 20.16 to 20.21 work well as assignment questions. Problem 20.20 is relatively challenging and requires students to have some programming skills.

QUESTIONS AND PROBLEMS

Problem 20.1.

Consider a position consisting of a $100,000 investment in asset A and a $100,000 investment in asset B. Assume that the daily volatilities of both assets are 1% and that the coefficient of correlation between their returns is 0.3. What is the 5-day 99% VaR for the portfolio?

The standard deviation of the daily change in the investment in each asset is $1,000. The variance of the portfolio's daily change is

\[ 1,000^2 + 1,000^2 + 2 \times 0.3 \times 1,000 \times 1,000 = 2,600,000 \]

The standard deviation of the portfolio's daily change is the square root of this or $1,612.45. The standard deviation of the 5-day change is

\[ 1,612.45 \times \sqrt{5} = $3,605.55 \]

From the tables of \( N(x) \) we see that \( N(-2.33) = 0.01 \). This means that 1% of a normal distribution lies more than 2.33 standard deviations below the mean. The 5-day 99 percent value at risk is therefore \( 2.33 \times 3,605.55 = $8,401 \).
Problem 20.2.

Describe three ways of handling interest-rate-dependent instruments when the model building approach is used to calculate VaR. How would you handle interest-rate-dependent instruments when historical simulation is used to calculate VaR?

The three alternative procedures mentioned in the chapter for handling interest rates when the model building approach is used to calculate VaR involve (a) the use of the duration model, (b) the use of cash flow mapping, and (c) the use of principal components analysis. When historical simulation is used we need to assume that the change in the zero-coupon yield curve between Day \( m \) and Day \( m + 1 \) is the same as that between Day \( i \) and Day \( i + 1 \) for different values of \( i \). In the case of a LIBOR, the zero curve is usually calculated from deposit rates, Eurodollar futures quotes, and swap rates. We can assume that the percentage change in each of these between Day \( m \) and Day \( m + 1 \) is the same as that between Day \( i \) and Day \( i + 1 \). In the case of a Treasury curve it is usually calculated from the yields on Treasury instruments. Again we can assume that the percentage change in each of these between Day \( m \) and Day \( m + 1 \) is the same as that between Day \( i \) and Day \( i + 1 \).

Problem 20.3.

A financial institution owns a portfolio of options on the U.S. dollar-sterling exchange rate. The delta of the portfolio is 56.0. The current exchange rate is 1.5000. Derive an approximate linear relationship between the change in the portfolio value and the percentage change in the exchange rate. If the daily volatility of the exchange rate is 0.7%, estimate the 10-day 99% VaR.

The approximate relationship between the daily change in the portfolio value, \( \Delta P \), and the daily change in the exchange rate, \( \Delta S \), is

\[
\Delta P = 56\Delta S
\]

The percentage daily change in the exchange rate, \( \Delta x \), equals \( \Delta S/1.5 \). It follows that

\[
\Delta P = 56 \times 1.5\Delta x
\]

or

\[
\Delta P = 84\Delta x
\]

The standard deviation of \( \Delta x \) equals the daily volatility of the exchange rate, or 0.7 percent. The standard deviation of \( \Delta P \) is therefore \( 84 \times 0.007 = 0.588 \). It follows that the 10-day 99 percent VaR for the portfolio is

\[
0.588 \times 2.33 \times \sqrt{10} = 4.33
\]

Problem 20.4.

Suppose you know that the gamma of the portfolio in the previous question is 16.2. How does this change your estimate of the relationship between the change in the portfolio value and the percentage change in the exchange rate?
The relationship is

\[ \Delta P = 56 \times 1.5 \Delta x + \frac{1}{2} \times 1.5^2 \times 16.2 \times \Delta x^2 \]

or

\[ \Delta P = 84 \Delta x + 18.225 \Delta x^2 \]

**Problem 20.5.**

Suppose that the daily change in the value of a portfolio is, to a good approximation, linearly dependent on two factors, calculated from a principal components analysis. The delta of a portfolio with respect to the first factor is 6 and the delta with respect to the second factor is -4. The standard deviations of the factor are 20 and 8, respectively. What is the 5-day 90% VaR?

The factors calculated from a principal components analysis are uncorrelated. The daily variance of the portfolio is

\[ 6^2 \times 20^2 + 4^2 \times 8^2 = 15,424 \]

and the daily standard deviation is \( \sqrt{15,424} = 124.19 \). Since \( N(-1.282) = 0.9 \), the 5-day 90% value at risk is

\[ 124.19 \times \sqrt{5} \times 1.282 = 356.01 \]

**Problem 20.6.**

Suppose a company has a portfolio consisting of positions in stocks, bonds, foreign exchange, and commodities. Assume there are no derivatives. Explain the assumptions underlying (a) the linear model and (b) the historical simulation model for calculating VaR.

The linear model assumes that the percentage daily change in each market variable has a normal probability distribution. The historical simulation model assumes that the probability distribution observed for the percentage daily changes in the market variables in the past is the probability distribution that will apply over the next day.

**Problem 20.7.**

Explain how an interest rate swap is mapped into a portfolio of zero-coupon bonds with standard maturities for the purposes of a VaR calculation.

When a final exchange of principal is added in, the floating side is equivalent a zero coupon bond with a maturity date equal to the date of the next payment. The fixed side is a coupon-bearing bond, which is equivalent to a portfolio of zero-coupon bonds. The swap can therefore be mapped into a portfolio of zero-coupon bonds with maturity dates corresponding to the payment dates. Each of the zero-coupon bonds can then be mapped into positions in the adjacent standard-maturity zero-coupon bonds.
Problem 20.8.

*Explain the difference between Value at Risk and Expected Shortfall.*

Value at risk is the loss that is expected to be exceeded \((100 - X)\%\) of the time in \(N\) days for specified parameter values, \(X\) and \(N\). Expected shortfall is the expected loss conditional that the loss is greater than the Value at Risk.

Problem 20.9.

*Explain why the linear model can provide only approximate estimates of VaR for a portfolio containing options.*

The change in the value of an option is not linearly related to the change in the value of the underlying variables. When the change in the values of underlying variables is normal, the change in the value of the option is non-normal. The linear model assumes that it is normal and is, therefore, only an approximation.

Problem 20.10.

*Verify that the 0.3-year zero-coupon bond in the cash-flow mapping example in the appendix to this chapter is mapped into a $37,397 position in a three-month bond and a $11,793 position in a six-month bond.*

The 0.3-year cash flow is mapped into a 3-month zero-coupon bond and a 6-month zero-coupon bond. The 0.25 and 0.50 year rates are 5.50 and 6.00 respectively. Linear interpolation gives the 0.30-year rate as 5.60%. The present value of $50,000 received at time 0.3 years is

\[
\frac{50,000}{1.056^{0.30}} = 49,189.32
\]

The volatility of 0.25-year and 0.50-year zero-coupon bonds are 0.06% and 0.10% per day respectively. The interpolated volatility of a 0.30-year zero-coupon bond is therefore 0.068% per day.

Assume that \(\alpha\) of the value of the 0.30-year cash flow gets allocated to a 3-month zero-coupon bond and \(1 - \alpha\) to a six-month zero coupon bond. To match variances we must have

\[
0.00068^2 = 0.0006^2\alpha^2 + 0.001^2(1 - \alpha)^2 + 2 \times 0.9 \times 0.0006 \times 0.001\alpha(1 - \alpha)
\]

or

\[
0.28\alpha^2 - 0.92\alpha + 0.5376 = 0
\]

Using the formula for the solution to a quadratic equation

\[
\alpha = \frac{-0.92 + \sqrt{0.92^2 - 4 \times 0.28 \times 0.5376}}{2 \times 0.28} = 0.760259
\]

this means that a value of 0.760259 \(\times 49,189.32 = \$37,397\) is allocated to the three-month bond and a value of 0.239741 \(\times 49,189.32 = \$11,793\) is allocated to the six-month bond.
The 0.3-year cash flow is therefore equivalent to a position of $37,397 in a 3-month zero-coupon bond and a position of $11,793 in a 6-month zero-coupon bond. This is consistent with the results in Table 20A.2 of the appendix to Chapter 20.

**Problem 20.11.**

Suppose that the 5-year rate is 6%, the seven year rate is 7% (both expressed with annual compounding), the daily volatility of a 5-year zero-coupon bond is 0.5%, and the daily volatility of a 7-year zero-coupon bond is 0.58%. The correlation between daily returns on the two bonds is 0.6. Map a cash flow of $1,000 received at time 6.5 years into a position in a five-year bond and a position in a seven-year bond using the approach in the appendix. What cash flows in five and seven years are equivalent to the 6.5-year cash flow?

The 6.5-year cash flow is mapped into a 5-year zero-coupon bond and a 7-year zero-coupon bond. The 5-year and 7-year rates are 6% and 7% respectively. Linear interpolation gives the 6.5-year rate as 6.75%. The present value of $1,000 received at time 6.5 years is

\[
\frac{1,000}{1.0675^{6.5}} = 654.05
\]

The volatility of 5-year and 7-year zero-coupon bonds are 0.50% and 0.58% per day respectively. The interpolated volatility of a 6.5-year zero-coupon bond is therefore 0.56% per day.

Assume that \( \alpha \) of the value of the 6.5-year cash flow gets allocated to a 5-year zero-coupon bond and \( 1 - \alpha \) to a 7-year zero-coupon bond. To match variances we must have

\[
.56^2 = .50^2\alpha^2 + .58^2(1 - \alpha)^2 + 2 \times 0.6 \times .50 \times .58\alpha(1 - \alpha)
\]

or

\[
.2384\alpha^2 - .3248\alpha + .0228 = 0
\]

Using the formula for the solution to a quadratic equation

\[
\alpha = \frac{.3248 - \sqrt{.3248^2 - 4 \times .2384 \times .0228}}{2 \times .2384}
\]

this means that a value of 0.074243 × 654.05 = $48.56 is allocated to the 5-year bond and a value of 0.925757 × 654.05 = $605.49 is allocated to the 7-year bond. The 6.5-year cash flow is therefore equivalent to a position of $48.56 in a 5-year zero-coupon bond and a position of $605.49 in a 7-year zero-coupon bond.

The equivalent 5-year and 7-year cash flows are 48.56 × 1.06^5 = 64.98 and 605.49 × 1.07^7 = 972.28.

**Problem 20.12.**

Some time ago a company has entered into a forward contract to buy £1 million for $1.5 million. The contract now has six months to maturity. The daily volatility of a six-month zero-coupon sterling bond (when its price is translated to dollars) is 0.06%
and the daily volatility of a six-month zero-coupon dollar bond is 0.05%. The correlation between returns from the two bonds is 0.8. The current exchange rate is 1.53. Calculate the standard deviation of the change in the dollar value of the forward contract in one day. What is the 10-day 99% VaR? Assume that the six-month interest rate in both sterling and dollars is 5% per annum with continuous compounding.

The contract is a long position in a sterling bond combined with a short position in a dollar bond. The value of the sterling bond is $1.53e^{-0.05 \times 0.5}$ or $1.492$ million. The value of the dollar bond is $1.5e^{-0.05 \times 0.5}$ or $1.463$ million. The variance of the change in the value of the contract in one day is

$$
1.492^2 \times 0.0006^2 + 1.463^2 \times 0.0005^2 - 2 \times 0.8 \times 1.492 \times 0.0006 \times 1.463 \times 0.0005
$$

$$
= 0.000000288
$$

The standard deviation is therefore $0.000537$ million. The 10-day 99% VaR is $0.000537 \times \sqrt{10} \times 2.33 = $0.00396 million.

Problem 20.13.

The text calculates a VaR estimate for the example in Table 20.5 assuming two factors. How does the estimate change if you assume (a) one factor and (b) three factors.

If we assume only one factor, the model is

$$
\Delta P = -0.08f_1
$$

The standard deviation of $f_1$ is 17.49. The standard deviation of $\Delta P$ is therefore $0.08 \times 17.49 = 1.40$ and the 1-day 99% value at risk is $1.40 \times 2.33 = 3.26$. If we assume three factors, our exposure to the third factor is

$$
10 \times (-0.37) + 4 \times (-0.38) - 8 \times (-0.30) - 7 \times (-0.12) + 2 \times (-0.04) = -2.06
$$

The model is therefore

$$
\Delta P = -0.08f_1 - 4.40f_2 - 2.06f_3
$$

The variance of $\Delta P$ is

$$
0.08^2 \times 17.49^2 + 4.40^2 \times 6.05^2 + 2.06^2 \times 3.10^2 = 751.36
$$

The standard deviation of $\Delta P$ is $\sqrt{751.36} = 27.41$ and the 1-day 99% value at risk is $27.41 \times 2.33 = $63.87.

The example illustrates that the relative importance of different factors depends on the portfolio being considered. Normally the second factor is less important than the first, but in this case it is much more important.

A bank has a portfolio of options on an asset. The delta of the options is -30 and the gamma is -5. Explain how these numbers can be interpreted. The asset price is 20 and its volatility per day is 1%. Adapt Sample Application E in the DerivaGem Application Builder software to calculate VaR.

The delta of the options is the rate if change of the value of the options with respect to the price of the asset. When the asset price increases by a small amount the value of the options decrease by 30 times this amount. The gamma of the options is the rate of change of their delta with respect to the price of the asset. When the asset price increases by a small amount, the delta of the portfolio decreases by five times this amount.

By entering 20 for \( S \), 1% for the volatility per day, -30 for delta, -5 for gamma, and recomputing we see that \( E(\Delta P) = -0.10, E(\Delta P^2) = 36.03, \) and \( E(\Delta P^3) = -32.415 \). The 1-day, 99% VaR given by the software for the quadratic approximation is 14.5. This is a 99% 1-day VaR. The VaR is calculated using the formulas in footnote 9 and the results in Technical Note 10.

Problem 20.15.

Suppose that in Problem 20.14 the vega of the portfolio is -2 per 1% change in the annual volatility. Derive a model relating the change in the portfolio value in one day to delta, gamma, and vega. Explain without doing detailed calculations how you would use the model to calculate a VaR estimate.

Define \( \sigma \) as the volatility per year, \( \Delta \sigma \) as the change in \( \sigma \) in one day, and \( \Delta w \) and the proportional change in \( \sigma \) in one day. We measure \( \sigma \) as a multiple of 1% so that the current value of \( \sigma \) is \( 1 \times \sqrt{252} = 15.87 \). The delta-gamma-vega model is

\[
\Delta P = -30\Delta S - 0.5 \times 5 \times (\Delta S)^2 - 2 \Delta \sigma 
\]

or

\[
\Delta P = -30 \times 20 \Delta x - 0.5 \times 5 \times 20^2 (\Delta x)^2 - 2 \times 15.87 \Delta w 
\]

which simplifies to

\[
\Delta P = -600 \Delta x - 1,000 (\Delta x)^2 - 31.74 \Delta w 
\]

The change in the portfolio value now depends on two market variables. Once the daily volatility of \( \sigma \) and the correlation between \( \sigma \) and \( S \) have been estimated we can estimate moments of \( \Delta P \) and use a Cornish–Fisher expansion.

ASSIGNMENT QUESTIONS

Problem 20.16.

A company has a position in bonds worth $6 million. The modified duration of the portfolio is 5.2 years. Assume that only parallel shifts in the yield curve can take place and that the standard deviation of the daily yield change (when yield is measured in percent) is 0.09. Use the duration model to estimate the 20-day 90% VaR for the portfolio. Explain
carefully the weaknesses of this approach to calculating VaR. Explain two alternatives that give more accuracy.

The change in the value of the portfolio for a small change $\Delta y$ in the yield is approximately $-DB\Delta y$ where $D$ is the duration and $B$ is the value of the portfolio. It follows that the standard deviation of the daily change in the value of the bond portfolio equals $DB\sigma_y$ where $\sigma_y$ is the standard deviation of the daily change in the yield. In this case $D = 5.2$, $B = 6,000,000$, and $\sigma_y = 0.0009$ so that the standard deviation of the daily change in the value of the bond portfolio is

$$5.2 \times 6,000,000 \times 0.0009 = 28,080$$

The 20-day 90% VaR for the portfolio is $1.282 \times 28,080 \times \sqrt{20} = 160,990$ or $\$160,990$. This approach assumes that only parallel shifts in the term structure can take place. Equivalently it assumes that all rates are perfectly correlated or that only one factor drives term structure movements. Alternative more accurate approaches described in the chapter are (a) cash flow mapping and (b) a principal components analysis.

Problem 20.17.

Consider a position consisting of a $300,000 investment in gold and a $500,000 investment in silver. Suppose that the daily volatilities of these two assets are 1.8% and 1.2% respectively, and that the coefficient of correlation between their returns is 0.6. What is the 10-day 97.5% VaR for the portfolio? By how much does diversification reduce the VaR?

The variance of the portfolio (in thousands of dollars) is

$$0.018^2 \times 300^2 + 0.012^2 \times 500^2 + 2 \times 300 \times 500 \times 0.6 \times 0.018 \times 0.012 = 104.04$$

The standard deviation is $\sqrt{104.04} = 10.2$. Since $N(-1.96) = 0.025$, the 1-day 97.5% VaR is $10.2 \times 1.96 = 19.99$ and the 10-day 97.5% VaR is $\sqrt{10} \times 19.99 = 63.22$. The 10-day 97.5% VaR is therefore $63,220$. The 10-day 97.5% value at risk for the gold investment is $5,400 \times \sqrt{10} \times 1.96 = 33,470$. The 10-day 97.5% value at risk for the silver investment is $6,000 \times \sqrt{10} \times 1.96 = 37,188$. The diversification benefit is

$$33,470 + 37,188 - 63,220 = \$7,438$$

Problem 20.18.

Consider a portfolio of options on a single asset. Suppose that the delta of the portfolio is 12, the value of the asset is $10$, and the daily volatility of the asset is 2%. Estimate the 1-day 95% VaR for the portfolio from the delta. Suppose next that the gamma of the portfolio is $-2.6$. Derive a quadratic relationship between the change in the portfolio value and the percentage change in the underlying asset price in one day. How would you use this in a Monte Carlo simulation?

An approximate relationship between the daily change in the value of the portfolio, $\Delta P$ and the proportional daily change in the value of the asset $\Delta x$ is

$$\Delta P = 10 \times 12 \Delta x = 120 \Delta x$$
The standard deviation of $\Delta x$ is 0.02. It follows that the standard deviation of $\Delta P$ is 2.4. The 1-day 95% VaR is $2.4 \times 1.65 = $3.96. Therefore, the standard deviation of $P$ is 2.4.

The 1-day 95% VaR is $2.4 \times 1.65 = $3.96. Therefore, the quadratic relationship is

$$\Delta P = 10 \times 12 \Delta x + 0.5 \times 10^2 \times (-2.6) \Delta x^2$$

or

$$\Delta P = 120 \Delta x - 130 \Delta x^2$$

this could be used in conjunction with Monte Carlo simulation. We would sample values for $\Delta x$ and use this equation to convert the $\Delta x$ samples to $\Delta P$ samples.

Problem 20.19.

A company has a long position in a two-year bond and a three-year bond as well as a short position in a five-year bond. Each bond has a principal of $100 and pays a 5% coupon annually. Calculate the company's exposure to the one-year, two-year, three-year, four-year, and five-year rates. Use the data in Tables 20.3 and 20.4 to calculate a 20 day 95% VaR on the assumption that rate changes are explained by (a) one factor, (b) two factors, and (c) three factors. Assume that the zero-coupon yield curve is flat at 5%.

The cash flows are as follows

<table>
<thead>
<tr>
<th>Year</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>2-yr Bond</td>
<td>5</td>
<td>105</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3-yr Bond</td>
<td>5</td>
<td>5</td>
<td>105</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5-yr Bond</td>
<td>-5</td>
<td>-5</td>
<td>-5</td>
<td>-5</td>
<td>-105</td>
</tr>
</tbody>
</table>

| Total  | 5    | 105  | 100  | -5   | -105 |
| Present Value | 4.756 | 95.008 | 86.071 | -4.094 | -81.774 |
| Impact of 1 bp change | -0.0005 | -0.0190 | -0.0258 | 0.0016 | 0.0409 |

The duration relationship is used to calculate the last row of the table. When the one-year rate increases by one basis point, the value of the cash flow in year 1 decreases by $1 \times 0.0001 \times 4.756 = 0.0005$; when the two year rate increases by one basis point, the value of the cash flow in year 2 decreases by $2 \times 0.0001 \times 95.008 = 0.0190$; and so on.

The sensitivity to the first factor is

$$-0.0005 \times 0.32 - 0.0190 \times 0.35 - 0.0258 \times 0.36 + 0.0016 \times 0.36 + 0.0409 \times 0.36$$

or $-0.00081$. The sensitivity to the second factor is

$$-0.0005 \times (-0.32) - 0.0190 \times (-0.10) - 0.0258 \times 0.02 + 0.0016 \times 0.14 + 0.0409 \times 0.17$$

or 0.0087. The sensitivity to the third factor is

$$-0.0005 \times (-0.37) - 0.0190 \times (-0.38) - 0.0258 \times (-0.30) + 0.0016 \times (-0.12)$$

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or 0.0133.

Assuming one factor, the standard deviation of the one-day change in the portfolio value is $0.00081 \times 17.49 = 0.0142$. The 20-day 95% VaR is therefore $0.0142 \times 1.645 \sqrt{20} = 0.104$

Assuming two factors, the standard deviation of the one-day change in the portfolio value is

$$\sqrt{0.00081^2 \times 17.49^2 + 0.0087^2 \times 6.05^2} = 0.0545$$

The 20-day 95% VaR is therefore $0.0545 \times 1.645 \sqrt{20} = 0.401$

Assuming three factors, the standard deviation of the one-day change in the portfolio value is

$$\sqrt{0.00081^2 \times 17.49^2 + 0.0087^2 \times 6.05^2 + 0.0133^2 \times 3.10^2} = 0.0683$$

The 20-day 95% VaR is therefore $0.0683 \times 1.645 \sqrt{20} = 0.502$.

In this case the second and third factor are important in calculating VaR.

**Problem 20.20.**

A bank has written a call option on one stock and a put option on another stock. For the first option the stock price is 50, the strike price is 51, the volatility is 28% per annum, and the time to maturity is nine months. For the second option the stock price is 20, the strike price is 19, the volatility is 25% per annum, and the time to maturity is one year. Neither stock pays a dividend, the risk-free rate is 6% per annum, and the correlation between stock price returns is 0.4. Calculate a 10-day 99% VaR

a. Using only deltas.
b. Using the partial simulation approach.
c. Using the full simulation approach.

This assignment is useful for consolidating students’ understanding of alternative approaches to calculating VaR, but it is calculation intensive. Realistically students need some programming skills to make the assignment feasible. My answer follows the usual practice of assuming that the 10-day 99% value at risk is $\sqrt{10}$ times the 1-day 99% value at risk. Some students may try to calculate a 10-day VaR directly, which is fine.

(a) From DerivaGem, the values of the two option positions are $-5.413$ and $-1.014$. The deltas are $-0.589$ and $0.284$, respectively. An approximate linear model relating the change in the portfolio value to proportional change, $\Delta x_1$, in the first stock price and the proportional change, $\Delta x_2$, in the second stock price is

$$\Delta P = -0.589 \times 50 \Delta x_1 + 0.284 \times 20 \Delta x_2$$

or

$$\Delta P = -29.45 \Delta x_1 + 5.68 \Delta x_2$$

The daily volatility of the two stocks are $0.28/\sqrt{252} = 0.0176$ and $0.25/\sqrt{252} = 0.0157$, respectively. The one-day variance of $\Delta P$ is

$$29.45^2 \times 0.0176^2 + 5.68^2 \times 0.0157^2 - 2 \times 29.45 \times 0.0176 \times 5.68 \times 0.0157 \times 0.4 = 0.2396$$
The one day standard deviation is, therefore, 0.4895 and the 10-day 99% VaR is 
2.33 \times \sqrt{10} \times 0.4895 = 3.61.

(b) In the partial simulation approach, we simulate changes in the stock prices over a one-
day period (building in the correlation) and then use the quadratic approximation to 
calculate the change in the portfolio value on each simulation trial. The one percentile 
point of the probability distribution of portfolio value changes turns out to be 1.22. 
The 10-day 99% value at risk is, therefore, 1.22\sqrt{10} or about 3.86.

(c) In the full simulation approach, we simulate changes in the stock price over one-day 
(building in the correlation) and revalue the portfolio on each simulation trial. The 
results are very similar to (b) and the estimate of the 10-day 99% value at risk is 
about 3.86.

Problem 20.21.

A common complaint of risk managers is that the model building approach (either 
linear or quadratic) does not work well when delta is close to zero. Test what happens 
when delta is close to zero in using Sample Application E in the DerivaGem Application 
Builder software. (You can do this by experimenting with different option positions and 
adjusting the position in the underlying to give a delta of zero.) Explain the results you 
get.

We can create a portfolio with zero delta in Sample Application E by changing the 
position in the stock from 1,000 to 513.58. (This reduces delta by 1,000−513.58 = 486.42.) In this case the true VaR is 48.86; the VaR given by the linear model is 0.00; and the VaR 
given by the quadratic model is -35.71.

Other zero-delta examples can be created by changing the option portfolio and then 
zeroing out delta by adjusting the position in the underlying asset. The results are similar. 
The software shows that neither the linear model nor the quadratic model gives good 
answers when delta is zero. The linear model always gives a VaR of zero because the 
model assumes that the portfolio has no risk. (For example, in the case of one underlying 
asset \( \Delta P = \Delta S \).) The quadratic model gives a negative VaR because \( \Delta P \) is always 
positive in this model. ( \( \Delta P = 0.5\Gamma(\Delta S)^2 \).)

In practice many portfolios do have deltas close to zero because of the hedging activ­
ities described in Chapter 17. This has led many financial institutions to prefer historical 
simulation to the model building approach.

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CHAPTER 21
Estimating Volatilities and Correlations

Notes for the Instructor

This chapter covers exponentially weighted moving average (EWMA) and GARCH (1,1) procedures for estimating the current level of a volatility or correlation. It explains maximum likelihood methods.

At the outset is important to make sure students understand the notation. The variable $\sigma_n$ is the volatility estimated for day $n$ at the end of day $n - 1$; $u_n$ is the realized return during day $n$. The EWMA approach, although not as sophisticated as GARCH(1,1), is widely used and is a useful lead-in to GARCH(1,1). After explaining both EWMA and GARCH (1,1) the chapter discusses maximum likelihood methods, the use of GARCH (1,1) for forecasting and the calculation of vega, and the application of the ideas to correlations.

Although there is readily available software for implementing GARCH (1,1) I like students to develop their own Excel applications. By doing this they develop a much better understanding of how maximum likelihood methods work. As indicated in the text the Solver routine in Excel is very effective if used in such a way that all the parameters being searched for are the same order of magnitude.

An interesting case study to teach in conjunction with the material in this chapter is Philippe Jorion’s one on Orange County. See http://www.gsm.uci.edu/~jorion/oc/case.html

Problems 21.15 to 21.18 can be used as assignment questions. They vary quite a bit in terms of the amount of time likely to be required. Problems 21.15 and 21.16 are a fairly quick test of whether students understand how EWMA and GARCH work. Problem 21.18 is somewhat longer. Problem 21.17 is longer again and requires some Excel skills.

QUESTIONS AND PROBLEMS.

Problem 21.1.

Explain the exponentially weighted moving average (EWMA) model for estimating volatility from historical data.

Define $u_i$ as $(S_i - S_{i-1})/S_{i-1}$, where $S_i$ is value of a market variable on day $i$. In the EWMA model, the variance rate of the market variable (i.e., the square of its volatility) calculated for day $n$ is a weighted average of the $u_{n-i}^2$'s ($i = 1, 2, 3, \ldots$). For some constant $\lambda$ ($0 < \lambda < 1$) the weight given to $u_{n-i}^2$ is $\lambda$ times the weight given to $u_{n-i-1}^2$. The volatility estimated for day $n$, $\sigma_n$, is related to the volatility estimated for day $n - 1$, $\sigma_{n-1}$, by

$$\sigma_n^2 = \lambda \sigma_{n-1}^2 + (1 - \lambda)u_{n-1}^2$$
This formula shows that the EWMA model has one very attractive property. To calculate the volatility estimate for day \( n \), it is sufficient to know the volatility estimate for day \( n - 1 \) and \( u_{n-1} \).

**Problem 21.2.**

*What is the difference between the exponentially weighted moving average model and the GARCH(1,1) model for updating volatilities?*

The EWMA model produces a forecast of the daily variance rate for day \( n \) which is a weighted average of (i) the forecast for day \( n - 1 \), and (ii) the square of the proportional change on day \( n - 1 \). The GARCH (1,1) model produces a forecast of the daily variance for day \( n \) which is a weighted average of (i) the forecast for day \( n - 1 \), (ii) the square of the proportional change on day \( n - 1 \) and (iii) a long run average variance rate. GARCH (1,1) adapts the EWMA model by giving some weight to a long run average variance rate. Whereas the EWMA has no mean reversion, GARCH (1,1) is consistent with a mean-reverting variance rate model.

**Problem 21.3.**

*The most recent estimate of the daily volatility of an asset is 1.5% and the price of the asset at the close of trading yesterday was $30.00. The parameter \( \lambda \) in the EWMA model is 0.94. Suppose that the price of the asset at the close of trading today is $30.50. How will this cause the volatility to be updated by the EWMA model?*

In this case \( \sigma_{n-1} = 0.015 \) and \( u_n = 0.5/30 = 0.01667 \), so that equation (21.7) gives

\[
\sigma_n^2 = 0.94 \times 0.015^2 + 0.06 \times 0.01667^2 = 0.0002281
\]

The volatility estimate on day \( n \) is therefore \( \sqrt{0.0002281} = 0.015103 \) or 1.5103%.

**Problem 21.4.**

*A company uses an EWMA model for forecasting volatility. It decides to change the parameter \( \lambda \) from 0.95 to 0.85. Explain the likely impact on the forecasts.*

Reducing \( \lambda \) from 0.95 to 0.85 means that more weight is put on recent observations of \( u_i^2 \) and less weight is given to older observations. Volatilities calculated with \( \lambda = 0.85 \) will react more quickly to new information and will “bounce around” much more than volatilities calculated with \( \lambda = 0.95 \).

**Problem 21.5.**

*The volatility of a certain market variable is 30% per annum. Calculate a 99% confidence interval for the size of the percentage daily change in the variable.*

The volatility per day is \( 30/\sqrt{252} = 1.89\% \). There is a 99% chance that a normally distributed variable will lie within 2.57 standard deviations. We are therefore 99% confident that the daily change will be less than \( 2.57 \times 1.89 = 4.86\% \).
Problem 21.6.
A company uses the GARCH(1,1) model for updating volatility. The three parameters are $\omega$, $\alpha$, and $\beta$. Describe the impact of making a small increase in each of the parameters while keeping the others fixed.

The weight given to the long-run average variance rate is $1 - \alpha - \beta$ and the long-run average variance rate is $\omega/(1 - \alpha - \beta)$. Increasing $\omega$ increases the long-run average variance rate; increasing $\alpha$ increases the weight given to the most recent data item, reduces the weight given to the long-run average variance rate, and increases the level of the long-run average variance rate. Increasing $\beta$ increases the weight given to the previous variance estimate, reduces the weight given to the long-run average variance rate, and increases the level of the long-run average variance rate.

Problem 21.7.
The most recent estimate of the daily volatility of the U.S. dollar-sterling exchange rate is 0.6% and the exchange rate at 4 p.m. yesterday was 1.5000. The parameter $\lambda$ in the EWMA model is 0.9. Suppose that the exchange rate at 4 p.m. today proves to be 1.4950. How would the estimate of the daily volatility be updated?

The proportional daily change is $-0.005/1.5000 = -0.003333$. The current daily variance estimate is $0.006^2 = 0.000036$. The new daily variance estimate is

$$0.9 \times 0.000036 + 0.1 \times 0.003333^2 = 0.000033511$$

The new volatility is the square root of this. It is 0.00579 or 0.579%.

Problem 21.8.
Assume that S&P 500 at close of trading yesterday was 1,040 and the daily volatility of the index was estimated as 1% per day at that time. The parameters in a GARCH(1,1) model are $\omega = 0.000002$, $\alpha = 0.06$, and $\beta = 0.92$. If the level of the index at close of trading today is 1,060, what is the new volatility estimate?

With the usual notation $u_{n-1} = 20/1040 = 0.01923$ so that

$$\sigma_n^2 = 0.000002 + 0.06 \times 0.01923^2 + 0.92 \times 0.01^2 = 0.0001162$$

so that $\sigma_n = 0.01078$. The new volatility estimate is therefore 1.078% per day.

Problem 21.9.
Suppose that the daily volatilities of asset A and asset B calculated at the close of trading yesterday are 1.6% and 2.5%, respectively. The prices of the assets at close of trading yesterday were $20 and $40 and the estimate of the coefficient of correlation between the returns on the two assets was 0.25. The parameter $\lambda$ used in the EWMA model is 0.95.

(a) Calculate the current estimate of the covariance between the assets.
(b) On the assumption that the prices of the assets at close of trading today are $20.5 and $40.5, update the correlation estimate.
(a) The volatilities and correlation imply that the current estimate of the covariance is
\[ 0.25 \times 0.016 \times 0.025 = 0.0001 \].

(b) If the prices of the assets at close of trading are $20.5 and $40.5, the proportional changes are 0.5/20 = 0.025 and 0.5/40 = 0.0125. The new covariance estimate is
\[ 0.95 \times 0.0001 + 0.05 \times 0.025 \times 0.0125 = 0.0001106 \].

The new variance estimate for asset A is
\[ 0.95 \times 0.016^2 + 0.05 \times 0.025^2 = 0.00027445 \]
so that the new volatility is 0.0166. The new variance estimate for asset B is
\[ 0.95 \times 0.025^2 + 0.05 \times 0.0125^2 = 0.000601562 \]
so that the new volatility is 0.0245. The new correlation estimate is
\[ \frac{0.0001106}{0.0166 \times 0.0245} = 0.272 \].

Problem 21.10.

The parameters of a GARCH(1,1) model are estimated as \( \omega = 0.000004 \), \( \alpha = 0.05 \), and \( \beta = 0.92 \). What is the long-run average volatility and what is the equation describing the way that the variance rate reverts to its long-run average? If the current volatility is 20% per year, what is the expected volatility in 20 days?

The long-run average variance rate is \( \omega/(1 - \alpha - \beta) \) or \( 0.000004/0.03 = 0.0001333 \). The long-run average volatility is \( \sqrt{0.0001333} \) or 1.155%. The equation describing the way the variance rate reverts to its long-run average is equation (21.13)

\[ E[\sigma^2_{n+k}] = V_L + (\alpha + \beta)^k(\sigma_n^2 - V_L) \]

In this case
\[ E[\sigma^2_{n+k}] = 0.0001333 + 0.97^k(\sigma_n^2 - 0.0001333) \]

If the current volatility is 20% per year, \( \sigma_n = 0.2/\sqrt{252} = 0.0126 \). The expected variance rate in 20 days is
\[ 0.0001333 + 0.97^{20}(0.0126^2 - 0.0001333) = 0.0001471 \]

The expected volatility in 20 days is therefore \( \sqrt{0.0001471} = 0.0121 \) or 1.21% per day.

Problem 21.11.

Suppose that the current daily volatilities of asset X and asset Y are 1.0% and 1.2%, respectively. The prices of the assets at close of trading yesterday were $30 and $50 and the estimate of the coefficient of correlation between the returns on the two assets made at
this time was 0.50. Correlations and volatilities are updated using a GARCH(1,1) model. The estimates of the model’s parameters are \( \alpha = 0.04 \) and \( \beta = 0.94 \). For the correlation \( \omega = 0.000001 \) and for the volatilities \( \omega = 0.000003 \). If the prices of the two assets at close of trading today are $31 and $51, how is the correlation estimate updated?

Using the notation in the text \( \sigma_{u,n-1} = 0.01 \) and \( \sigma_{v,n-1} = 0.012 \) and the most recent estimate of the covariance between the asset returns is \( \text{cov}_{n-1} = 0.01 \times 0.012 \times 0.50 = 0.00006 \). The variable \( u_{n-1} = 1/30 = 0.03333 \) and the variable \( v_{n-1} = 1/50 = 0.02 \). The new estimate of the covariance, \( \text{cov}_n \), is

\[
0.000001 + 0.04 \times 0.03333 \times 0.02 + 0.94 \times 0.000006 = 0.0000841
\]

The new estimate of the variance of the first asset, \( \sigma_{u,n}^2 \) is

\[
0.000003 + 0.04 \times 0.03333^2 + 0.94 \times 0.01^2 = 0.0001414
\]

so that \( \sigma_{u,n} = \sqrt{0.0001414} = 0.01189 \) or 1.189%. The new estimate of the variance of the second asset, \( \sigma_{v,n}^2 \) is

\[
0.000003 + 0.04 \times 0.02^2 + 0.94 \times 0.012^2 = 0.0001544
\]

so that \( \sigma_{v,n} = \sqrt{0.0001544} = 0.01242 \) or 1.242%. The new estimate of the correlation between the assets is therefore \( \frac{0.0000841}{(0.01189 \times 0.01242)} = 0.569 \).

**Problem 21.12.**

Suppose that the daily volatility of the FT-SE 100 stock index (measured in pounds sterling) is 1.8% and the daily volatility of the dollar/sterling exchange rate is 0.9%. Suppose further that the correlation between the FT-SE 100 and the dollar/sterling exchange rate is 0.4. What is the volatility of the FT-SE 100 when it is translated to U.S. dollars? Assume that the dollar/sterling exchange rate is expressed as the number of U.S. dollars per pound sterling. (Hint: When \( Z = XY \), the percentage daily change in \( Z \) is approximately equal to the percentage daily change in \( X \) plus the percentage daily change in \( Y \).)

The FT-SE expressed in dollars is \( XY \) where \( X \) is the FT-SE expressed in sterling and \( Y \) is the exchange rate (value of one pound in dollars). Define \( x_i \) as the proportional change in \( X \) on day \( i \) and \( y_i \) as the proportional change in \( Y \) on day \( i \). The proportional change in \( XY \) is approximately \( x_i + y_i \). The standard deviation of \( x_i \) is 0.018 and the standard deviation of \( y_i \) is 0.009. The correlation between the two is 0.4. The variance of \( x_i + y_i \) is therefore

\[
0.018^2 + 0.009^2 + 2 \times 0.018 \times 0.009 \times 0.4 = 0.0005346
\]

so that the volatility of \( x_i + y_i \) is 0.0231 or 2.31%. This is the volatility of the FT-SE expressed in dollars. Note that it is greater than the volatility of the FT-SE expressed in sterling. This is the impact of the positive correlation. When the FT-SE increases the
value of sterling measured in dollars also tends to increase. This creates an even bigger increase in the value of FT-SE measured in dollars. Similarly for a decrease in the FT-SE.

Suppose that in Problem 21.12 the correlation between the S&P 500 Index (measured in dollars) and the FT-SE 100 Index (measured in sterling) is 0.7, the correlation between the S&P 500 index (measured in dollars) and the dollar-sterling exchange rate is 0.3, and the daily volatility of the S&P 500 Index is 1.6%. What is the correlation between the S&P 500 Index (measured in dollars) and the FT-SE 100 Index when it is translated to dollars? (Hint: For three variables $X$, $Y$, and $Z$, the covariance between $X + Y$ and $Z$ equals the covariance between $X$ and $Z$ plus the covariance between $Y$ and $Z$.)

Continuing with the notation in Problem 21.12, define $z_i$ as the proportional change in the value of the S&P 500 on day $i$. The covariance between $x_i$ and $z_i$ is $0.7 \times 0.018 \times 0.016 = 0.0002016$. The covariance between $y_i$ and $z_i$ is $0.3 \times 0.009 \times 0.016 = 0.0000432$. The covariance between $x_i + y_i$ and $z_i$ equals the covariance between $x_i$ and $z_i$ plus the covariance between $y_i$ and $z_i$. It is

$$0.0002016 + 0.0000432 = 0.0002448$$

The correlation between $x_i + y_i$ and $z_i$ is

$$\frac{0.0002448}{0.016 \times 0.0231} = 0.662$$

Note that the volatility of the S&P 500 drops out in this calculation.

Show that the GARCH (1,1) model

$$\sigma^2_n = \omega + \alpha u^2_{n-1} + \beta \sigma^2_{n-1}$$

in equation (21.9) is equivalent to the stochastic volatility model

$$dV = a(V_L - V) dt + \xi V dz$$

where time is measured in days and $V$ is the square of the volatility of the asset price and

$$a = 1 - \alpha - \beta$$

$$V_L = \frac{\omega}{1 - \alpha - \beta}$$

$$\xi = \alpha \sqrt{2}$$

What is the stochastic volatility model when time is measure in years?
(Hint: The variable \( u_{n-1} \) is the return on the asset price in time \( \Delta t \). It can be assumed to be normally distributed with mean zero and standard deviation \( \sigma_{n-1} \). It follows that the mean of \( u_{n-1}^2 \) and \( u_{n-1}^4 \) are \( \sigma_{n-1}^2 \) and \( 3\sigma_{n-1}^4 \), respectively.)

\[
\sigma_n^2 = \omega + \alpha u_{n-1}^2 + \beta \sigma_{n-1}^2
\]

so that

\[
\sigma_n^2 - \sigma_{n-1}^2 = \omega + (\beta - 1)\sigma_{n-1}^2 + \alpha u_{n-1}^2
\]

The variable \( u_{n-1}^2 \) has a mean of \( \sigma_{n-1}^2 \) and a variance of

\[
E(u_{n-1})^4 - [E(u_{n-1}^2)]^2 = 2\sigma_{n-1}^4
\]

The standard deviation of \( u_{n-1}^2 \) is \( \sqrt{2}\sigma_{n-1}^2 \). Assuming the \( u_i \) are generated by a Wiener process, \( dz \), we can therefore write

\[
u_{n-1}^2 = \sigma_{n-1}^2 + \sqrt{2}\sigma_{n-1}^2 \epsilon
\]

where \( \epsilon \) is a random sample from a standard normal distribution. Substituting this into the equation for \( \sigma_n^2 - \sigma_{n-1}^2 \) we get

\[
\sigma_n^2 - \sigma_{n-1}^2 = \omega + (\alpha + \beta - 1)\sigma_{n-1}^2 + \alpha \sqrt{2}\sigma_{n-1}^2 \epsilon
\]

We can write \( \Delta V = \sigma_n^2 - \sigma_{n-1}^2 \) and \( V = \sigma_{n-1}^2 \). Also \( \alpha = 1 - \alpha - \beta \), \( aV_L = \omega \), and \( \xi = \alpha \sqrt{2} \) so that

\[
\Delta V = a(V_L - V) + \xi V \epsilon
\]

Because time is measured in days, \( \Delta t = 1 \) and

\[
\Delta V = a(V_L - V)\Delta t + \xi V \epsilon \sqrt{\Delta t}
\]

The result follows.

When time is measured in years \( \Delta t = 1/252 \) so that

\[
\Delta V = a(V_L - V)252\Delta t + \xi V \epsilon \sqrt{252\Delta t}
\]

and the process for \( V \) is

\[
dV = 252a(V_L - V) \, dt + \xi V \sqrt{252} \, dz
\]

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ASSIGNMENT QUESTIONS

Problem 21.15.
Suppose that the price of gold at close of trading yesterday was $600 and its volatility was estimated as 1.3% per day. The price at the close of trading today is $596. Update the volatility estimate using
(a) The EWMA model with $\lambda = 0.94$
(b) The GARCH(1,1) model with $\omega = 0.000002$, $\alpha = 0.04$, and $\beta = 0.94$.

The proportional change in the price of gold is $-4/600 = -0.00667$. Using the EWMA model the variance is updated to

$$0.94 \times 0.013^2 + 0.06 \times 0.00667^2 = 0.00016153$$

so that the new daily volatility is $\sqrt{0.00016153} = 0.01271$ or 1.271% per day. Using GARCH (1,1) the variance is updated to

$$0.000002 + 0.94 \times 0.013^2 + 0.04 \times 0.00667^2 = 0.00016264$$

so that the new daily volatility is $\sqrt{0.00016264} = 0.1275$ or 1.275% per day.

Problem 21.16.
Suppose that in Problem 21.15 the price of silver at the close of trading yesterday was $16, its volatility was estimated as 1.5% per day, and its correlation with gold was estimated as 0.8. The price of silver at the close of trading today is unchanged at $16. Update the volatility of silver and the correlation between silver and gold using the two models in Problem 21.15. In practice, is the $\omega$ parameter likely to be the same for gold and silver?

The proportional change in the price of silver is zero. Using the EWMA model the variance is updated to

$$0.94 \times 0.015^2 + 0.06 \times 0 = 0.0002115$$

so that the new daily volatility is $\sqrt{0.0002115} = 0.1454$ or 1.454% per day. Using GARCH (1,1) the variance is updated to

$$0.000002 + 0.94 \times 0.015^2 + 0.04 \times 0 = 0.0002135$$

so that the new daily volatility is $\sqrt{0.0002135} = 0.1461$ or 1.461% per day. The initial covariance is $0.8 \times 0.013 \times 0.015 = 0.000156$ Using EWMA the covariance is updated to

$$0.94 \times 0.000156 + 0.06 \times 0 = 0.00014664$$

so that the new correlation is $0.00014664/(0.01454 \times 0.1271) = 0.7934$ Using GARCH (1,1) the covariance is updated to

$$0.000002 + 0.94 \times 0.000156 + 0.04 \times 0 = 0.00014864$$

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so that the new correlation is $0.00014864/(0.01461 \times 0.01275) = 0.7977$.

For a given $\alpha$ and $\beta$, the $\omega$ parameter defines the long run average value of a variance or a covariance. There is no reason why we should expect the long run average daily variance for gold and silver should be the same. There is also no reason why we should expect the long run average covariance between gold and silver to be the same as the long run average variance of gold or the long run average variance of silver. In practice, therefore, we are likely to want to allow $\omega$ in a GARCH(1,1) model to vary from market variable to market variable. (Some instructors may want to use this problem as a lead in to multivariate GARCH models.

**Problem 21.17.**

An Excel spreadsheet containing over 900 days of daily data on a number of different exchange rates and stock indices can be downloaded from the author’s website: http://www.rotman.utoronto.ca/ hull. Choose one exchange rate and one stock index. Estimate the value of $\lambda$ in the EWMA model that minimizes the value

$$\sum_i (v_i - \beta_i)^2$$

where $v_i$ is the variance forecast made at the end of day $i - 1$ and $\beta_i$ is the variance calculated from data between day $i$ and day $i + 25$. Use the Solver tool in Excel. Set the variance forecast at the end of the first day equal to the square of the return on that day to start the EWMA calculations.

My results give “best” values for $\lambda$ higher than the 0.94 used by RiskMetrics. For AUD, BEF, CHF, DEM, DKK, ESP, FRF, GBP, ITL, NLG, and SEK they are 0.983, 0.967, 0.968, 0.960, 0.971, 0.983, 0.965, 0.977, 0.939, 0.962, and 0.989, respectively. For TSE, S&P, FTSE, CAC, and Nikkei, they are 0.991, 0.989, 0.958, 0.974, and 0.961, respectively.

**Problem 21.18.**

Suppose that the parameters in a GARCH (1,1) model are $\alpha = 0.03$, $\beta = 0.95$ and $\omega = 0.000002$.

a. What is the long-run average volatility?
b. If the current volatility is 1.5% per day, what is your estimate of the volatility in 20, 40, and 60 days?
c. What volatility should be used to price 20-, 40-, and 60-day options?
d. Suppose that there is an event that increases the current volatility by 0.5% to 2% per day. Estimate the effect on the volatility in 20, 40, and 60 days.
e. Estimate by how much does the event increase the volatilities used to price 20-, 40-, and 60-day options?

(a) The long-run average variance, $V_L$, is

$$\frac{\omega}{1 - \alpha - \beta} = \frac{0.000002}{0.02} = 0.0001$$

The long run average volatility is $\sqrt{0.0001} = 0.01$ or 1% per day
(b) From equation (2.13) the expected variance in 20 days is

\[ 0.0001 + 0.98^{20}(0.015^2 - 0.0001) = 0.000183 \]

The expected volatility per day is therefore \( \sqrt{0.000183} = 0.0135 \) or 1.35%. Similarly the expected volatilities in 40 and 60 days are 1.25% and 1.17%, respectively.

(c) In equation (2.14) \( a = \ln(1/0.98) = 0.0202 \) the variance used to price options in 20 days is

\[ 252(0.00001 + \frac{1 - e^{-0.0202 \times 20}}{0.0202 \times 20})(0.01566 - 0.00001) = 0.051 \]

so that the volatility per annum is 22.61%. Similarly, the volatilities that should be used for 40- and 60-day options are 21.63% and 20.85% per annum, respectively.

(d) From equation (2.13) the expected variance in 20 days is

\[ 0.0001 + 0.98^{20}(0.02^2 - 0.0001) = 0.0003 \]

The expected volatility per day is therefore \( \sqrt{0.0003} = 0.0135 \) or 1.73%. Similarly the expected volatilities in 40 and 60 days are 1.53% and 1.38% per day, respectively.

(e) When today's volatility increases from 1.5% per day (23.81% per year) to 2% per day (31.75% per year) the equation (2.15) gives the 20-day volatility increase as

\[ \frac{1 - e^{-0.0202 \times 20}}{0.0202 \times 20} \times \frac{23.81}{22.61} \times (31.75 - 23.81) = 6.88 \]

or 6.88% bringing the volatility up to 29.49%. Similarly the 40- and 60-day volatilities increase to 27.37% and 25.70%. A more exact calculation using equation (2.14) gives 29.56%, 27.76%, and 26.27% as the three volatilities.
CHAPTER 22
Credit Risk

Notes for the Instructor

This chapter covers the quantification of credit risk and prepares the way for the material in Chapter 23 on credit derivatives. It has been updated and improved for the seventh edition. After explaining credit ratings I spend some time talking about the difference between real world (physical) default probabilities and risk-neutral (implied) default probabilities. As Tables 22.4 and 22.5 show the difference between the two is much higher than might be expected with the risk-neutral default probabilities being higher. However, the extra expected return of bond traders as a result of this is not excessive. The important point to emphasize is that bonds do not default independently of each other and, as a result, there is systematic risk that is priced in the market.

Section 22.6 provides an outline of Merton's model for implying default probabilities from equity prices and how it is used in practice. Section 22.7 outlines how derivatives transactions should be adjusted for credit risk. Section 22.8 discusses netting, collateralization agreements and downgrade triggers. Section 22.9 introduces the Gaussian copula model which is used in both Basel II (see Business Snapshot 22.2) and in the valuation of credit derivatives (see Chapter 23).

I generally allow two classes for this chapter and two classes for Chapter 23. Problem 22.30 works well for class discussion. Problems 22.28, 22.29, 22.31, and 22.32 work well as assignments. Some Excel skills are necessary for Problems 22.28 and 22.32.

QUESTIONS AND PROBLEMS

Problem 22.1.

The spread between the yield on a three-year corporate bond and the yield on a similar risk-free bond is 50 basis points. The recovery rate is 30%. Estimate the average default intensity per year over the three-year period.

From equation (22.2) the average default intensity over the three years is $0.0050/(1-0.3) = 0.0071$ or 0.71% per year.

Problem 22.2.

Suppose that in Problem 22.1 the spread between the yield on a five-year bond issued by the same company and the yield on a similar risk-free bond is 60 basis points. Assume the same recovery rate of 30%. Estimate the average default intensity per year over the five-year period. What do your results indicate about the average default intensity in years 4 and 5?
From equation (22.2) the average default intensity over the five years is \( \frac{0.0060}{1 - 0.3} = 0.0086 \) or 0.86% per year. Using the results in the previous question, the default intensity is 0.71% per year for the first three years and
\[
\frac{0.0086 \times 5 - 0.0071 \times 3}{2} = 0.0107
\]
or 1.07% per year in years 4 and 5.

**Problem 22.3.**

*Should researchers use real-world or risk-neutral default probabilities for a) calculating credit value at risk and b) adjusting the price of a derivative for defaults?*

Real-world probabilities of default should be used for calculating credit value at risk. Risk-neutral probabilities of default should be used for adjusting the price of a derivative for default.

**Problem 22.4.**

*How are recovery rates usually defined?*

The recovery rate for a bond is the value of the bond immediately after the issuer defaults as a percent of its face value.

**Problem 22.5.**

*Explain the difference between an unconditional default probability density and a default intensity.*

The default intensity, \( h(t) \) at time \( t \) is defined so that \( h(t)\Delta t \) is the probability of default between times \( t \) and \( t + \Delta t \) conditional on no default prior to time \( t \). The unconditional default probability density \( q(t) \) is defined so that \( q(t)\Delta t \) is the probability of default between times \( t \) and \( t + \Delta t \) as seen at time zero.

**Problem 22.6.**

*Verify a) that the numbers in the second column of Table 22.4 are consistent with the numbers in Table 22.1 and b) that the numbers in the fourth column of Table 22.5 are consistent with the numbers in Table 22.4 and a recovery rate of 40%.*

The first number in the second column of Table 22.4 is calculated as
\[
-\frac{1}{7} \ln(1 - 0.00251) = 0.000359
\]
or 0.04% per year. Other numbers in the column are calculated similarly. The numbers in the fourth column of Table 22.5 are the numbers in the second column of Table 22.4 multiplied by one minus the expected recovery rate. In this case the expected recovery rate is 0.4.
Problem 22.7.

Describe how netting works. A bank already has one transaction with a counterparty on its books. Explain why a new transaction by a bank with a counterparty can have the effect of increasing or reducing the bank's credit exposure to the counterparty.

Suppose company A goes bankrupt when it has a number of outstanding contracts with company B. Netting means that the contracts with a positive value to A are netted against those with a negative value in order to determine how much, if anything, company A owes company B. Company A is not allowed to "cherry pick" by keeping the positive-value contracts and defaulting on the negative-value contracts.

The new transaction will increase the bank's exposure to the counterparty if the contract tends to have a positive value whenever the existing contract has a positive value and a negative value whenever the existing contract has a negative value. However, if the new transaction tends to offset the existing transaction, it is likely to have the incremental effect of reducing credit risk.

Problem 22.8.

Suppose that the measure $\beta_{AB}(T)$ in equation (22.9) is the same in the real world and the risk-neutral world. Is the same true of the Gaussian copula measure, $\rho_{AB}$?

Equation (22.14) gives the relationship between $\beta_{AB}(T)$ and $\rho_{AB}$. This involves $Q_A(T)$ and $Q_B(T)$. These change as we move from the real world to the risk-neutral world. It follows that the relationship between $\beta_{AB}(T)$ and $\rho_{AB}$ in the real world is not the same as in the risk-neutral world. If $\beta_{AB}(T)$ is the same in the two worlds, $\rho_{AB}$ is not.

Problem 22.9.

What is meant by a haircut in a collateralization agreement. A company offers to post its own equity as collateral. How would you respond?

When securities are pledged as collateral the haircut is the discount applied to their market value for margin calculations. A company’s own equity would not be good collateral. When the company defaults on its contracts its equity is likely to be worth very little.

Problem 22.10.

Explain the difference between the Gaussian copula model for the time to default and CreditMetrics as far as the following are concerned: a) the definition of a credit loss and b) the way in which default correlation is modeled.

(a) In the Gaussian copula model for time to default a credit loss is recognized only when a default occurs. In CreditMetrics it is recognized when there is a credit downgrade as well as when there is a default.

(b) In the Gaussian copula model of time to default, the default correlation arises because the value of the factor $M$. This defines the default environment or average default rate in the economy. In CreditMetrics a copula model is applied to credit ratings migration and this determines the joint probability of particular changes in the credit ratings of two companies.
Problem 22.11.

Suppose that the probability of company A defaulting during a two year period is 0.2 and the probability of company B defaulting during this period is 0.15. If the Gaussian copula measure of default correlation is 0.3, what is the binomial correlation measure?

In equation (22.14), \( Q_A(2) = 0.2, Q_B(2) = 0.15, \) and \( \rho_{AB} = 0.3. \) Also

\[
\begin{align*}
x_A(2) &= N^{-1}(0.2) = -0.84162 \\
x_B(2) &= N^{-1}(0.15) = -1.03643 \\
M(-0.84162, -1.03643, 0.3) &= 0.0522 \\
\beta_{AB}(2) &= \frac{0.0522 - 0.2 \times 0.15}{\sqrt{(0.2 - 0.2^2)(0.15 - 0.15^2)}} = 0.156
\end{align*}
\]

Problem 22.12.

Suppose that the LIBOR/swap curve is flat at 6% with continuous compounding and a five-year bond with a coupon of 5% (paid semiannually) sells for 90.00. How would an asset swap on the bond be structured? What is the asset swap spread that would be calculated in this situation?

Suppose that the principal is $100. The asset swap is structured so that the $10 is paid initially. After that $2.50 is paid every six months. In return LIBOR plus a spread is received on the principal of $100. The present value of the fixed payments is

\[
10 + 2.5e^{-0.06 \times 0.5} + 2.5e^{-0.06 \times 1} + \ldots + 2.5e^{-0.06 \times 5} + 100e^{-0.06 \times 5} = 105.3579
\]

The spread over LIBOR must therefore have a present value of 5.3579. The present value of $1 received every six months for five years is 8.5105. The spread received every six months must therefore be 5.3579/8.5105 = $0.6296. The asset swap spread is therefore 2 x 0.6296 = 1.2592% per annum.

Problem 22.13.

Show that the value of a coupon-bearing corporate bond is the sum of the values of its constituent zero-coupon bonds when the amount claimed in the event of default is the no-default value of the bond, but that this is not so when the claim amount is the face value of the bond plus accrued interest.

When the claim amount is the no-default value, the loss for a corporate bond arising from a default at time \( t \) is

\[
v(t)(1 - \hat{R})B^*
\]

where \( v(t) \) is the discount factor for time \( t \) and \( B^* \) is the no-default value of the bond at time \( t \). Suppose that the zero-coupon bonds comprising the corporate bond have no-default values at time \( t \) of \( Z_1, Z_2, \ldots, Z_n \), respectively. The loss from the \( i \)th zero-coupon bond arising from a default at time \( t \) is

\[
v(t)(1 - \hat{R})Z_i
\]
The total loss from all the zero-coupon bonds is

\[ v(t)(1 - \hat{R}) \sum_{i=1}^{n} Z_i = v(t)(1 - \hat{R})B^* \]

This shows that the loss arising from a default at time \( t \) is the same for the corporate bond as for the portfolio of its constituent zero-coupon bonds. It follows that the value of the corporate bond is the same as the value of its constituent zero-coupon bonds.

When the claim amount is the face value plus accrued interest, the loss for a corporate bond arising from a default at time \( t \) is

\[ v(t)B^* - v(t)\hat{R}[L + a(t)] \]

where \( L \) is the face value and \( a(t) \) is the accrued interest at time \( t \). In general this is not the same as the loss from the sum of the losses on the constituent zero-coupon bonds.

**Problem 22.14.**

A four-year corporate bond provides a coupon of 4% per year payable semiannually and has a yield of 5% expressed with continuous compounding. The risk-free yield curve is flat at 3% with continuous compounding. Assume that defaults can take place at the end of each year (immediately before a coupon or principal payment and the recovery rate is 30%. Estimate the risk-neutral default probability on the assumption that it is the same each year.

Define \( Q \) as the risk-free rate. The calculations are as follows

<table>
<thead>
<tr>
<th>Time (yrs)</th>
<th>Def. Prob.</th>
<th>Recovery Amount ($)</th>
<th>Risk-free Value ($)</th>
<th>Loss Given Default ($)</th>
<th>Discount Factor</th>
<th>PV of Expected Loss ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>( Q )</td>
<td>30</td>
<td>104.78</td>
<td>74.78</td>
<td>0.9704</td>
<td>72.57Q</td>
</tr>
<tr>
<td>2.0</td>
<td>( Q )</td>
<td>30</td>
<td>103.88</td>
<td>73.88</td>
<td>0.9418</td>
<td>69.58Q</td>
</tr>
<tr>
<td>3.0</td>
<td>( Q )</td>
<td>30</td>
<td>102.96</td>
<td>72.96</td>
<td>0.9139</td>
<td>66.68Q</td>
</tr>
<tr>
<td>4.0</td>
<td>( Q )</td>
<td>30</td>
<td>102.00</td>
<td>72.00</td>
<td>0.8869</td>
<td>63.86Q</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>272.69Q</td>
</tr>
</tbody>
</table>

The bond pays a coupon of 2 every six months and has a continuously compounded yield of 5% per year. Its market price is 96.19. The risk-free value of the bond is obtained by discounting the promised cash flows at 3%. It is 103.66. The total loss from defaults should therefore be equated to 103.66 - 96.19 = 7.46. The value of \( Q \) implied by the bond price is therefore given by 272.69\( Q \) = 7.46. or \( Q = 0.0274 \). The implied probability of default is 2.74% per year.

**Problem 22.15.**

A company has issued 3- and 5-year bonds with a coupon of 4% per annum payable annually. The yields on the bonds (expressed with continuous compounding) are 4.5%
and 4.75%, respectively. Risk-free rates are 3.5% with continuous compounding for all maturities. The recovery rate is 40%. Defaults can take place half way through each year. The risk-neutral default rates per year are $Q_1$ for years 1 to 3 and $Q_2$ for years 4 and 5. Estimate $Q_1$ and $Q_2$.

The table for the first bond is

<table>
<thead>
<tr>
<th>Time (yrs)</th>
<th>Def.</th>
<th>Recovery Amount ($)</th>
<th>Risk-free Value ($)</th>
<th>Loss Given Default ($)</th>
<th>Discount Factor</th>
<th>PV of Expected Loss ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>$Q_1$</td>
<td>40</td>
<td>103.01</td>
<td>63.01</td>
<td>0.9827</td>
<td>61.92$Q_1$</td>
</tr>
<tr>
<td>1.5</td>
<td>$Q_1$</td>
<td>40</td>
<td>102.61</td>
<td>62.61</td>
<td>0.9489</td>
<td>59.41$Q_1$</td>
</tr>
<tr>
<td>2.5</td>
<td>$Q_1$</td>
<td>40</td>
<td>102.20</td>
<td>62.20</td>
<td>0.9162</td>
<td>56.98$Q_1$</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>178.31$Q_1$</td>
</tr>
</tbody>
</table>

The market price of the bond is 98.35 and the risk-free value is 101.23. It follows that $Q_1$ is given by

$$178.31Q_1 = 101.23 - 98.35$$

so that $Q_1 = 0.0161$.

The table for the second bond is

<table>
<thead>
<tr>
<th>Time (yrs)</th>
<th>Def.</th>
<th>Recovery Amount ($)</th>
<th>Risk-free Value ($)</th>
<th>Loss Given Default ($)</th>
<th>Discount Factor</th>
<th>PV of Expected Loss ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>$Q_1$</td>
<td>40</td>
<td>103.77</td>
<td>63.77</td>
<td>0.9827</td>
<td>62.67$Q_1$</td>
</tr>
<tr>
<td>1.5</td>
<td>$Q_1$</td>
<td>40</td>
<td>103.40</td>
<td>63.40</td>
<td>0.9489</td>
<td>60.16$Q_1$</td>
</tr>
<tr>
<td>2.5</td>
<td>$Q_1$</td>
<td>40</td>
<td>103.01</td>
<td>63.01</td>
<td>0.9162</td>
<td>57.73$Q_1$</td>
</tr>
<tr>
<td>3.5</td>
<td>$Q_2$</td>
<td>40</td>
<td>102.61</td>
<td>62.61</td>
<td>0.8847</td>
<td>55.39$Q_2$</td>
</tr>
<tr>
<td>4.5</td>
<td>$Q_2$</td>
<td>40</td>
<td>102.20</td>
<td>62.20</td>
<td>0.8543</td>
<td>53.13$Q_2$</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>180.56$Q_1 + 108.53Q_2$</td>
</tr>
</tbody>
</table>

The market price of the bond is 96.24 is and the risk-free value is 101.97. It follows that

$$180.56Q_1 + 108.53Q_2 = 101.97 - 96.24$$

From which we get $Q_2 = 0.0260$ The bond prices therefore imply a probability of default of 1.61% per year for the first three years and 2.60% for the next two years.

**Problem 22.16.**

Suppose that a financial institution has entered into a swap dependent on the sterling interest rate with counterparty X and an exactly offsetting swap with counterparty Y. Which of the following statements are true and which are false.
(a) The total present value of the cost of defaults is the sum of the present value of the cost of defaults on the contract with X plus the present value of the cost of defaults on the contract with Y.

(b) The expected exposure in one year on both contracts is the sum of the expected exposure on the contract with X and the expected exposure on the contract with Y.

(c) The 95% upper confidence limit for the exposure in one year on both contracts is the sum of the 95% upper confidence limit for the exposure in one year on the contract with X and the 95% upper confidence limit for the exposure in one year on the contract with Y.

Explain your answers.

The statements in (a) and (b) are true. The statement in (c) is not. Suppose that \( v_X \) and \( v_Y \) are the exposures to X and Y. The expected value of \( v_X + v_Y \) is the expected value of \( v_X \) plus the expected value of \( v_Y \). The same is not true of 95% confidence limits.

Problem 22.17.

A company enters into a one-year forward contract to sell $100 for AUD150. The contract is initially at the money. In other words, the forward exchange rate is 1.50. The one-year dollar risk-free rate of interest is 5% per annum. The one-year dollar rate of interest at which the counterparty can borrow is 6% per annum. The exchange rate volatility is 12% per annum. Estimate the present value of the cost of defaults on the contract? Assume that defaults are recognized only at the end of the life of the contract.

The cost of defaults is \( uv \) where \( u \) is percentage loss from defaults during the life of the contract and \( v \) is the value of an option that pays off \( \max(150S_T - 100, 0) \) in one year and \( S_T \) is the value in dollars of one AUD. The value of \( u \) is

\[
u = 1 - e^{-0.06} = 0.009950\]

The variable \( v \) is 150 times a call option to buy one AUD for 0.6667. The formula for the call option in terms of forward prices is

\[
[FN(d_1) - KN(d_2)]e^{-rT}
\]

where

\[
d_1 = \frac{\log(F/K) + \sigma^2T/2}{\sigma\sqrt{T}}
\]

\[
d_2 = d_1 - \sigma\sqrt{T}
\]

In this case \( F = 0.6667, K = 0.6667, \sigma = 0.12, T = 1, \) and \( r = 0.05 \) so that \( d_1 = 0.06, d_2 = -0.06 \) and the value of the call option is 0.0303. It follows that \( v = 150 \times 0.0303 = 4.545 \) so that the cost of defaults is

\[
4.545 \times 0.009950 = 0.04522
\]
Problem 22.18.

Suppose that in Problem 22.17, the six-month forward rate is also 1.50 and the six-month dollar risk-free interest rate is 5% per annum. Suppose further that the six-month dollar rate of interest at which the counterparty can borrow is 5.5% per annum. Estimate the present value of the cost of defaults assuming that defaults can occur either at the six-month point or at the one-year point? (If a default occurs at the six-month point, the company’s potential loss is the market value of the contract.)

In this case the costs of defaults is $u_1 v_1 + u_2 v_2$ where

$$u_1 = 1 - e^{-(0.055-0.05) \times 0.5} = 0.002497$$

$$u_2 = e^{-(0.055-0.05) \times 0.5} - e^{-(0.06-0.05) \times 1} = 0.007453$$

$v_1$ is the value of an option that pays off $\max(150S_T - 100, 0)$ in six months and $v_2$ is the value of a option that pays off $\max(150S_T - 100, 0)$ in one year. The calculations in Problem 22.17 shows that $v_2$ is 4.545. Similarly $v_1 = 3.300$ so that the cost of defaults is

$$0.002497 \times 3.300 + 0.007453 \times 4.545 = 0.04211$$

Problem 22.19.

"A long forward contract subject to credit risk is a combination of a short position in a no-default put and a long position in a call subject to credit risk." Explain this statement.

Assume that defaults happen only at the end of the life of the forward contract. In a default-free world the forward contract is the combination of a long European call and a short European put where the strike price of the options equals the delivery price and the maturity of the options equals the maturity of the forward contract. If the no-default value of the contract is positive at maturity, the call has a positive value and the put is worth zero. The impact of defaults on the forward contract is the same as that on the call. If the no-default value of the contract is negative at maturity, the call has a zero value and the put has a positive value. In this case defaults have no effect. Again the impact of defaults on the forward contract is the same as that on the call. It follows that the contract has a value equal to a long position in a call that is subject to default risk and short position in a default-free put.

Problem 22.20.

Explain why the credit exposure on a matched pair of forward contracts resembles a straddle.

Suppose that the forward contract provides a payoff at time $T$. With our usual notation, the value of a long forward contract is $S_T - Ke^{-rT}$. The credit exposure on a long forward contract is therefore $\max(S_T - Ke^{-rT}, 0)$; that is, it is a call on the asset price with strike price $Ke^{-rT}$. Similarly the credit exposure on a short forward contract is $\max(Ke^{-rT} - S_T, 0)$; that is, it is a put on the asset price with strike price $Ke^{-rT}$. The total credit exposure is, therefore, a straddle with strike price $Ke^{-rT}$.
Problem 22.21.

Explain why the impact of credit risk on a matched pair of interest rate swaps tends to be less than that on a matched pair of currency swaps.

The credit risk on a matched pair of interest rate swaps is $|B_{\text{fixed}} - B_{\text{floating}}|$. As maturity is approached all bond prices tend to par and this tends to zero. The credit risk on a matched pair of currency swaps is $|SB_{\text{foreign}} - B_{\text{fixed}}|$ where $S$ is the exchange rate. The expected value of this tends to increase as the swap maturity is approached because of the uncertainty in $S$.

Problem 22.22.

"When a bank is negotiating currency swaps, it should try to ensure that it is receiving the lower interest rate currency from a company with a low credit risk.” Explain.

As time passes there is a tendency for the currency which has the lower interest rate to strengthen. This means that a swap where we are receiving this currency will tend to move in the money (i.e., have a positive value). Similarly a swap where we are paying the currency will tend to move out of the money (i.e., have a negative value). From this it follows that our expected exposure on the swap where we are receiving the low-interest currency is much greater than our expected exposure on the swap where we are receiving the high-interest currency. We should therefore look for counterparties with a low credit risk on the side of the swap where we are receiving the low-interest currency. On the other side of the swap we are far less concerned about the creditworthiness of the counterparty.

Problem 22.23.

Does put–call parity hold when there is default risk? Explain your answer.

No, put–call parity does not hold when there is default risk. Suppose $c^*$ and $p^*$ are the no-default prices of a European call and put with strike price $K$ and maturity $T$ on a non-dividend-paying stock whose price is $S$, and that $c$ and $p$ are the corresponding values when there is default risk. The text shows that when we make the independence assumption (that is, we assume that the variables determining the no-default value of the option are independent of the variables determining default probabilities and recovery rates), $c = c^* e^{-\nu(T)-\nu^*(T)T}$ and $p = p^* e^{-\nu(T)-\nu^*(T)T}$. The relationship

$$c^* + Ke^{-\nu^*(T)T} = p^* + S$$

which holds in a no-default world therefore becomes

$$c + Ke^{-\nu(T)T} = p + Se^{-\nu(T)-\nu^*(T)T}$$

when there is default risk. This is not the same a regular put–call parity. What is more, the relationship depends on the independence assumption and cannot be deduced from the same sort of simple no-arbitrage arguments that we used in Chapter 9 for the put–call parity relationship in a no-default world.
Problem 22.24.

Suppose that in an asset swap $B$ is the market price of the bond per dollar of principal, $B^*$ is the default-free value of the bond per dollar of principal, and $V$ is the present value of the asset swap spread per dollar of principal. Show that $V = B^* - B$.

We can assume that the principal is paid and received at the end of the life of the swap without changing the swap's value. If the spread were zero the present value of the floating payments per dollar of principal would be 1. The payment of LIBOR plus the spread therefore has a present value of $1 + V$. The payment of the bond cash flows has a present value per dollar of principal of $B^*$. The initial payment required from the payer of the bond cash flows per dollar of principal is $1 - B$. (This may be negative; an initial amount of $B - 1$ is then paid by the payer of the floating rate). Because the asset swap is initially worth zero we have

$$1 + V = B^* + 1 - B$$

so that

$$V = B^* - B$$

Problem 22.25.

Show that under Merton's model in Section 22.6 the credit spread on a $T$-year zero-coupon bond is $- \ln[N(d_2) + N(-d_1)/L]/T$ where $L = De^{-rT}/V_0$.

The value of the debt in Merton's model is $V_0 - E_0$ or

$$De^{-rT}N(d_2) - V_0 N(d_1) + V_0 = De^{-rT}N(d_2) + V_0 N(-d_1)$$

If the credit spread is $s$ this should equal $De^{-(r+s)T}$ so that

$$De^{-(r+s)T} = De^{-rT}N(d_2) + V_0 N(-d_1)$$

Substituting $De^{-rT} = LV_0$

$$LV_0 e^{-sT} = LV_0 N(d_2) + V_0 N(-d_1)$$

or

$$Le^{-sT} = LN(d_2) + N(-d_1)$$

so that

$$s = - \ln[N(d_2) + N(-d_1)/L]/T$$


Suppose that the spread between the yield on a 3-year zero-coupon riskless bond and a 3-year zero-coupon bond issued by a corporation is 1%. By how much does Black–Scholes overstate the value of a 3-year European option sold by the corporation.

When the default risk of the seller of the option is taken into account the option value is the Black–Scholes price multiplied by $e^{-0.01\times3} = 0.9704$. Black–Scholes overprices the option by about 3%.
Problem 22.27

Give an example of a) right-way risk and b) wrong-way risk.

(a) Right way risk describes the situation when a default by the counterparty is most likely to occur when the contract has a positive value to the counterparty. An example of right way risk would be when a counterparty's future depends on the price of a commodity and it enters into a contract to partially hedging that exposure.

(b) Wrong way risk describes the situation when a default by the counterparty is most likely to occur when the contract has a negative value to the counterparty. An example of wrong way risk would be when a counterparty is a speculator and the contract has the same exposure as the rest of the counterparty's portfolio.

ASSIGNMENT QUESTIONS

Problem 22.28.

Suppose a three-year corporate bond provides a coupon of 7% per year payable semi-annually and has a yield of 5% (expressed with semiannual compounding). The yields for all maturities on risk-free bonds is 4% per annum (expressed with semiannual compounding). Assume that defaults can take place every six months (immediately before a coupon payment) and the recovery rate is 45%. Estimate the default probabilities assuming a) the unconditional default probabilities are the same on each possible default date and b) assuming that the default probabilities conditional on no earlier default are the same on each possible default date.

(a) The market price of the bond is 105.51. The risk-free price is 108.40. The expected cost of defaults is therefore 2.89. The following table calculates the cost of defaults as 348.20Q where Q is the unconditional probability of default each year. This means that the probability of default per year is 2.89/348.20 or 0.00831.

<table>
<thead>
<tr>
<th>Time (yrs)</th>
<th>Def. Prob.</th>
<th>Recovery Amount ($)</th>
<th>Risk-free Value ($)</th>
<th>Loss Given Default ($)</th>
<th>Discount Factor</th>
<th>PV of Expected Loss ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>Q</td>
<td>45</td>
<td>110.57</td>
<td>65.57</td>
<td>0.9804</td>
<td>64.28Q</td>
</tr>
<tr>
<td>1.0</td>
<td>Q</td>
<td>45</td>
<td>109.21</td>
<td>64.21</td>
<td>0.9612</td>
<td>61.73Q</td>
</tr>
<tr>
<td>1.5</td>
<td>Q</td>
<td>45</td>
<td>107.83</td>
<td>62.83</td>
<td>0.9423</td>
<td>59.20Q</td>
</tr>
<tr>
<td>2.0</td>
<td>Q</td>
<td>45</td>
<td>106.41</td>
<td>61.41</td>
<td>0.9238</td>
<td>56.74Q</td>
</tr>
<tr>
<td>2.5</td>
<td>Q</td>
<td>45</td>
<td>104.97</td>
<td>59.97</td>
<td>0.9057</td>
<td>54.32Q</td>
</tr>
<tr>
<td>3.0</td>
<td>Q</td>
<td>45</td>
<td>103.50</td>
<td>58.50</td>
<td>0.8880</td>
<td>51.95Q</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>348.20Q</td>
</tr>
</tbody>
</table>

(b) Suppose that Q* is the default probability conditional on no earlier default. The unconditional default probabilities in 0.5, 1.0, 1.5, 2.0, 2.5, 3.0 years are Q*, Q*(1 − Q*), Q*(1 − Q*)², Q*(1 − Q*)³, Q*(1 − Q*)⁴, Q*(1 − Q*)⁵. We must therefore find
the value of $Q^*$ that solves

$$64.28Q^* + 61.73Q^*(1 - Q^*) + 59.20 * Q^*(1 - Q^*)^2 + 56.74Q^*(1 - Q^*)^3$$

$$+ 54.32Q^*(1 - Q^*)^4 + 51.95Q^*(1 - Q^*)^5 = 2.89$$

Using Solver in Excel we find that $Q^* = 0.00848$.

**Problem 22.29.**

A company has one- and two-year bonds outstanding, each providing a coupon of 8% per year payable annually. The yields on the bonds (expressed with continuous compounding) are 6.0% and 6.6%, respectively. Risk-free rates are 4.5% for all maturities. The recovery rate is 35%. Defaults can take place half way through each year. Estimate the risk-neutral default rate each year.

Consider the first bond. Its market price is $108e^{-0.06\times1} = 101.71$. Its default-free price is $108e^{-0.045\times1} = 103.25$. The present value of the loss from defaults is therefore 1.54. In this case losses can take place at only one time, half way through the year. Suppose that the probability of default at this time is $Q_1$. The default-free value of the bond is $108e^{-0.045\times0.5} = 105.60$. The loss in the event of a default is $105.60 - 35 = 70.60$. The present value of the expected loss is $70.60e^{-0.045\times0.5}Q_1 = 69.03Q_1$. It follows that

$$69.03Q_1 = 1.54$$

so that $Q_1 = 0.0223$.

Now consider the second bond. Its market price is 103.32 and its default-free value is 106.35. The present value of the loss from defaults is therefore 3.03. At time 0.5 the default free value of the bond is 108.77. The loss in the event of a default is therefore 73.77. The present value of the loss from defaults at this time is $72.13Q_1$ or 1.61. This means that the present value of the loss from defaults at the 1.5 year point is 3.03 - 1.61 or 1.42. The default-free value of the bond at the 1.5 year point is 105.60. The loss in the event of a default is 70.60. The present value of the expected loss is $65.99Q_2$ where $Q_2$ is the probability of default in the second year. It follows that

$$65.99Q_2 = 1.42$$

so that $Q_2 = 0.0216$.

The probabilities of default in years one and two are therefore 2.23% and 2.16%.

**Problem 22.30.**

Explain carefully the distinction between real-world and risk-neutral default probabilities. Which is higher? A bank enters into a credit derivative where it agrees to pay $100 at the end of one year if a certain company's credit rating falls from A to Baa or lower during the year. The one-year risk-free rate is 5%. Using Table 22.6, estimate a value for the derivative. What assumptions are you making? Do they tend to overstate or understate the value of the derivative.
Real world default probabilities are the true probabilities of defaults. They can be estimated from historical data. Risk-neutral default probabilities are the probabilities of defaults in a world where all market participants are risk neutral. They can be estimated from bond prices. Risk-neutral default probabilities are higher. This means that returns in the risk-neutral world are lower. From Table 22.6 the probability of a company moving from A to Baa or lower in one year is 5.73%. An estimate of the value of the derivative is therefore $0.0573 \times 100 \times e^{-0.05 \times 1} = 5.45$. The approximation in this is that we are using the real-world probability of a downgrade. To value the derivative correctly we should use the risk-neutral probability of a downgrade. Since the risk-neutral probability of a default is higher than the real-world probability, it seems likely that the same is true of a downgrade. This means that 5.45 is likely to be too low as an estimate of the value of the derivative.

Problem 22.31.

The value of a company’s equity is $4 million and the volatility of its equity is 60%. The debt that will have to be repaid in two years is $15 million. The risk-free interest rate is 6% per annum. Use Merton’s model to estimate the expected loss from default, the probability of default, and the recovery rate in the event of default. Explain why Merton’s model gives a high recovery rate. (Hint The Solver function in Excel can be used for this question.)

In this case $E_0 = 4$, $\sigma_E = 0.60$, $D = 15$, $r = 0.06$. Setting up the data in Excel, we can solve equations (22.3) and (22.4) by using the approach in footnote 12. The solution to the equations proves to be $V_0 = 17.084$ and $\sigma_V = 0.1576$. The probability of default is $N(-d_2)$ or 15.61%. The market value of the debt is 17.084 - 4 = 13.084. The present value of the promised payment on the debt is $15e^{-0.06 \times 2} = 13.304$. The expected loss on the debt is, therefore, $(13.304 - 13.084)/13.304$ or 1.65% of its no-default value. The expected recovery rate in the event of default is therefore $(15.61 - 1.65)/15.61$ or about 89%. The reason the recovery rate is so high is as follows. There is a default if the value of the assets moves from 17.08 to below 15. A value for the assets significantly below 15 is unlikely. Conditional on a default, the expected value of the assets is, therefore, not a huge amount below 15. In practice it is likely that companies manage to delay defaults until asset values are well below the face value of the debt.

Problem 22.32.

Suppose that a bank has a total of $10 million of exposures of a certain type. The one-year probability of default averages 1% and the recovery rate averages 40%. The copula correlation parameter is 0.2. Estimate the 99.5% one-year credit VaR.

From equation (22.11) the 99.5% worst case probability of default is

$$N \left( \frac{N^{-1}(0.01) + \sqrt{0.2}N^{-1}(0.995)}{\sqrt{0.8}} \right) = 0.0946$$

This gives the 99.5% credit VaR as $10 \times (1 - 0.4) \times 0.0946 = 0.568$ millions of dollars or $568,000$. 

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CHAPTER 23
Credit Derivatives

Notes for the Instructor

Chapter 23 has been significantly revised for the seventh edition. Part of the chapter is now devoted to asset-backed securities and a discussion of the credit crunch of 2007. This comes before the material on CDOs. There is now more material (with numerical examples) on the use of the one-factor Gaussian copula model to value CDOs.

Credit derivatives are an exciting and relatively recent innovation. They have the potential to allow organizations to manage their credit risks in much the same way that they manage market risks. Students generally enjoy the material in Chapter 23. Indeed many quickly decide that they would love to work in the credit derivatives area because of the pace of innovation.

This chapter follows on from Chapter 22. It starts by discussing in some detail how credit default swaps work and how they are valued. It then moves on to binary CDSs, CDS forwards and options, total return swaps, basket credit default swaps, asset-backed securities, and collateralized debt obligations. The section on convertible bonds has been moved to Chapter 26.

Problem 23.27, 23.28, and 23.29 can be discussed in class. Others work well as assignment questions. Problems 23.30 and 23.31 are more challenging.

QUESTIONS AND PROBLEMS

Problem 23.1.

Explain the difference between a regular credit default swap and a binary credit default swap.

Both provide insurance against a particular company defaulting during a period of time. In a credit default swap the payoff is the notional principal amount multiplied by one minus the recovery rate. In a binary swap the payoff is the notional principal.

Problem 23.2.

A credit default swap requires a semiannual payment at the rate of 60 basis points per year. The principal is $300 million and the credit default swap is settled in cash. A default occurs after four years and two months, and the calculation agent estimates that the price of the cheapest deliverable bond is 40% of its face value shortly after the default. List the cash flows and their timing for the seller of the credit default swap.

The seller receives

\[ 300,000,000 \times 0.0060 \times 0.5 = 900,000 \]
at times 0.5, 1.0, 1.5, 2.0, 2.5, 3.0, 3.5, and 4.0 years. The seller also receives a final accrual payment of about $300,000 (= $300,000,000 \times 0.060 \times 2/12) at the time of the default (4 years and two months). The seller pays

\[ 300,000,000 \times 0.6 = 180,000,000 \]

at the time of the default.

**Problem 23.3.**

*Explain the two ways a credit default swap can be settled.*

Sometimes there is physical settlement and sometimes there is cash settlement. In the event of a default when there is physical settlement the buyer of protection sells bonds issued by the reference entity for their face value. Bonds with a total face value equal to the notional principal can be sold. In the event of a default when there is cash settlement a calculation agent estimates the value of the cheapest-to-deliver bonds issued by the reference entity a specified number of days after the default event. The cash payoff is then based on the excess of the face value of these bonds over the estimated value.

**Problem 23.4.**

*Explain how a CDO and a Synthetic CDO are created.*

A CDO is created from a bond portfolio. The returns from the bond portfolio flow to a number of tranches (i.e., different categories of investors). The tranches differ as far as the credit risk they assume. The first tranche might have an investment in 5% of the bond portfolio and be responsible for the first 5% of losses. The next tranche might have an investment in 10% of the portfolio and be responsible for the next 10% of the losses, and so on. In a synthetic CDO there is no bond portfolio. Instead a portfolio of credit default swaps is sold and the resulting credit risks are allocated to tranches in a similar way to that just described.

**Problem 23.5.**

*Explain what a first-to-default credit default swap is. Does its value increase or decrease as the default correlation between the companies in the basket increases? Explain.*

In a first-to-default basket CDS there are a number of reference entities. When the first one defaults there is a payoff (calculated in the usual way for a CDS) and basket CDS terminates. The value of a first-to-default basket CDS decreases as the correlation between the reference entities in the basket increases. This is because the probability of a default is high when the correlation is zero and decreases as the correlation increases. In the limit when the correlation is one there is in effect only one company and the probability of a default is quite low.

**Problem 23.6.**

*Explain the difference between risk-neutral and real-world default probabilities.*

Risk-neutral default probabilities are backed out from credit default swaps or bond prices. Real-world default probabilities are calculated from historical data.
Problem 23.7.

*Explain why a total return swap can be useful as a financing tool.*

Suppose a company wants to buy some assets. If a total return swap is used, a financial institution buys the assets and enters into a swap with the company where it pays the company the return on the assets and receives from the company LIBOR plus a spread. The financial institution has less risk than it would have if it lent the company money and used the assets as collateral. This is because, in the event of a default by the company it owns the assets.

Problem 23.8.

*Suppose that the risk-free zero curve is flat at 7% per annum with continuous compounding and that defaults can occur half way through each year in a new five-year credit default swap. Suppose that the recovery rate is 30% and the default probabilities each year conditional on no earlier default is 3% Estimate the credit default swap spread. Assume payments are made annually.*

The table corresponding to Tables 23.1, giving unconditional default probabilities, is

<table>
<thead>
<tr>
<th>Time (years)</th>
<th>Default Probability</th>
<th>Survival Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0300</td>
<td>0.9700</td>
</tr>
<tr>
<td>2</td>
<td>0.0291</td>
<td>0.9409</td>
</tr>
<tr>
<td>3</td>
<td>0.0282</td>
<td>0.9127</td>
</tr>
<tr>
<td>4</td>
<td>0.0274</td>
<td>0.8853</td>
</tr>
<tr>
<td>5</td>
<td>0.0266</td>
<td>0.8587</td>
</tr>
</tbody>
</table>

The table corresponding to Table 23.2, giving the present value of the expected regular payments (payment rate is $s$ per year), is

<table>
<thead>
<tr>
<th>Time (years)</th>
<th>Probability of Survival</th>
<th>Expected Payment</th>
<th>Discount Factor</th>
<th>PV of Expected Payment</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.9700</td>
<td>0.9700$s$</td>
<td>0.9324</td>
<td>0.9044$s$</td>
</tr>
<tr>
<td>2</td>
<td>0.9409</td>
<td>0.9409$s$</td>
<td>0.8694</td>
<td>0.8180$s$</td>
</tr>
<tr>
<td>3</td>
<td>0.9127</td>
<td>0.9127$s$</td>
<td>0.8106</td>
<td>0.7398$s$</td>
</tr>
<tr>
<td>4</td>
<td>0.8853</td>
<td>0.8853$s$</td>
<td>0.7558</td>
<td>0.6691$s$</td>
</tr>
<tr>
<td>5</td>
<td>0.8587</td>
<td>0.8587$s$</td>
<td>0.7047</td>
<td>0.6051$s$</td>
</tr>
</tbody>
</table>

Total: $3.7364$s

The table corresponding to Table 23.3, giving the present value of the expected payoffs (notional principal = $1$), is

301
The table corresponding to Table 23.4, giving the present value of accrual payments, is:

<table>
<thead>
<tr>
<th>Time (years)</th>
<th>Probability of Default</th>
<th>Recovery Rate</th>
<th>Expected Payoff</th>
<th>Discount Factor</th>
<th>PV of Expected Payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.0300</td>
<td>0.3</td>
<td>0.0210</td>
<td>0.9656</td>
<td>0.0203</td>
</tr>
<tr>
<td>1.5</td>
<td>0.0291</td>
<td>0.3</td>
<td>0.0204</td>
<td>0.9003</td>
<td>0.0183</td>
</tr>
<tr>
<td>2.5</td>
<td>0.0282</td>
<td>0.3</td>
<td>0.0198</td>
<td>0.8395</td>
<td>0.0166</td>
</tr>
<tr>
<td>3.5</td>
<td>0.0274</td>
<td>0.3</td>
<td>0.0192</td>
<td>0.7827</td>
<td>0.0150</td>
</tr>
<tr>
<td>4.5</td>
<td>0.0266</td>
<td>0.3</td>
<td>0.0186</td>
<td>0.7298</td>
<td>0.0136</td>
</tr>
</tbody>
</table>

Total: 0.0838

The credit default swap spread $s$ is given by:

$$3.7364s + 0.0598s = 0.0838$$

It is 0.0221 or 221 basis points.

**Problem 23.9.**

*What is the value of the swap in Problem 23.8 per dollar of notional principal to the protection buyer if the credit default swap spread is 150 basis points?*

If the credit default swap spread is 150 basis points, the value of the swap to the buyer of protection is:

$$0.0838 - (3.7364 + 0.0598) \times 0.0150 = 0.0269$$

per dollar of notional principal.

**Problem 23.10.**

*What is the credit default swap spread in Problem 23.8 if it is a binary CDS?*

If the swap is a binary CDS, the present value of expected payoffs is calculated as follows:
The credit default swap spread $s$ is given by:

$$3.7364s + 0.05988s = 0.1197$$

It is 0.0315 or 315 basis points.

**Problem 23.11.**

How does a five-year $n$th-to-default credit default swap work? Consider a basket of 100 reference entities where each reference entity has a probability of defaulting in each year of 1%. As the default correlation between the reference entities increases what would you expect to happen to the value of the swap when a) $n = 1$ and b) $n = 25$. Explain your answer.

A five-year $n$th to default credit default swap works in the same way as a regular credit default swap except that there is a basket of companies. The payoff occurs when the $n$th default from the companies in the basket occurs. After the $n$th default has occurred the swap ceases to exist. When $n = 1$ (so that the swap is a “first to default”) an increase in the default correlation lowers the value of the swap. When the default correlation is zero there are 100 independent events that can lead to a payoff. As the correlation increases the probability of a payoff decreases. In the limit when the correlation is perfect there is in effect only one company and therefore only one event that can lead to a payoff.

When $n = 25$ (so that the swap is a 25th to default) an increase in the default correlation increases the value of the swap. When the default correlation is zero there is virtually no chance that there will be 25 defaults and the value of the swap is very close to zero. As the correlation increases the probability of multiple defaults increases. In the limit when the correlation is perfect there is in effect only one company and the value of a 25th-to-default credit default swap is the same as the value of a first-to-default swap.
Problem 23.12.
What is the formula relating the payoff on a CDS to the notional principal and the recovery rate?

The payoff is \( L(1 - R) \) where \( L \) is the notional principal and \( R \) is the recovery rate.

Problem 23.13.
Show that the spread for a new plain vanilla CDS should be \((1 - R)\) times the spread for a similar new binary CDS where \( R \) is the recovery rate.

The payoff from a plain vanilla CDS is \(1 - R\) times the payoff from a binary CDS with the same principal. The payoff always occurs at the same time on the two instruments. It follows that the regular payments on a new plain vanilla CDS must be \(1 - R\) times the payments on a new binary CDS. Otherwise there would be an arbitrage opportunity.

Verify that if the CDS spread for the example in Tables 23.1 to 21.4 is 100 basis points and the probability of default in a year (conditional on no earlier default) must be 1.61%. How does the probability of default change when the recovery rate is 20% instead of 40%? Verify that your answer is consistent with the implied probability of default being approximately proportional to \(1/(1 - R)\) where \( R \) is the recovery rate.

The 1.61% implied default probability can be calculated by setting up a worksheet in Excel and using Solver. To verify that 1.61% is correct we note that, with a conditional default probability of 1.61%, the unconditional probabilities are:

<table>
<thead>
<tr>
<th>Time (years)</th>
<th>Default Probability</th>
<th>Survival Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0161</td>
<td>0.9839</td>
</tr>
<tr>
<td>2</td>
<td>0.0158</td>
<td>0.9681</td>
</tr>
<tr>
<td>3</td>
<td>0.0156</td>
<td>0.9525</td>
</tr>
<tr>
<td>4</td>
<td>0.0153</td>
<td>0.9371</td>
</tr>
<tr>
<td>5</td>
<td>0.0151</td>
<td>0.9221</td>
</tr>
</tbody>
</table>

The present value of the regular payments becomes 4.11708, the present value of the expected payoffs becomes 0.0415, and the present value of the expected accrual payments becomes 0.03465. When \( s = 0.01 \) the present value of the expected payments equals the present value of the expected payoffs.

When the recovery rate is 20% the implied default probability (calculated using Solver) is 1.21% per year. Note that 1.21/1.61 is approximately equal to \((1-0.4)/(1-0.2)\) showing that the implied default probability is approximately proportional to \(1/(1 - R)\).

In passing we note that if the CDS spread is used to imply an unconditional default probability (assumed to be the same each year) then this implied unconditional default probability is exactly proportional to \(1/(1 - R)\). When we use the CDS spread to imply a conditional default probability (assumed to be the same each year) it is only approximately proportional to \(1/(1 - R)\).
Problem 23.15.

A company enters into a total return swap where it receives the return on a corporate bond paying a coupon of 5% and pays LIBOR. Explain the difference between this and a regular swap where 5% is exchanged for LIBOR.

In the case of a total return swap a company receives (pays) the increase (decrease) in the value of the bond. In the regular swap this does not happen.

Problem 23.16.

Explain how forward contracts and options on credit default swaps are structured.

When a company enters into a long (short) forward contract it is obligated to buy (sell) the protection given by a specified credit default swap with a specified spread at a specified future time. When a company buys a call (put) option contract it has the option to buy (sell) the protection given by a specified credit default swap with a specified spread at a specified future time. Both contracts are normally structured so that they cease to exist if a default occurs during the life of the contract.

Problem 23.17.

"The position of a buyer of a credit default swap is similar to the position of someone who is long a risk-free bond and short a corporate bond." Explain this statement.

A credit default swap insures a corporate bond issued by the reference entity against default. Its approximate effect is to convert the corporate bond into a risk-free bond. The buyer of a credit default swap has therefore chosen to exchange a corporate bond for a risk-free bond. This means that the buyer is long a risk-free bond and short a similar corporate bond.

Problem 23.18.

Why is there a potential asymmetric information problem in credit default swaps?

Payoffs from credit default swaps depend on whether a particular company defaults. Arguably some market participants have more information about this that other market participants. (See Business Snapshot 23.2.)

Problem 23.19.

Does valuing a CDS using real-world default probabilities rather than risk-neutral default probabilities overstate or understate its value? Explain your answer.

Real world default probabilities are less than risk-neutral default probabilities. It follows that the use of actuarial default probabilities will tend to understate the value of a CDS.

Problem 23.20.

What is the difference between a total return swap and an asset swap?

In an asset swap the bond’s promised payments are swapped for LIBOR plus a spread. In a total return swap the bond’s actual payments are swapped for LIBOR plus a spread.
Problem 23.21.
Suppose that in a one-factor Gaussian copula model the five-year probability of default for each of 125 names is 3% and the pairwise copula correlation is 0.2. Calculate, for factor values of –2, –1, 0, 1, and 2, a) the default probability conditional on the factor value and b) the probability of more than 10 defaults conditional on the factor value.

Using equation (23.2) the probability of default conditional on a factor value of $M$ is

$$N \left( \frac{N^{-1}(0.03) - \sqrt{0.2M}}{\sqrt{1 - 0.2}} \right)$$

For $M$ equal to –2, –1, 0, 1, and 2 the probabilities of default are 0.135, 0.054, 0.018, 0.005, and 0.001 respectively. To six decimal places the probability of more than 10 defaults for these values of $M$ can be calculated using the BINOMDIST function in Excel. They are 0.959284, 0.79851, 0.000016, 0, and 0, respectively.

Problem 23.22.

Explain the difference between base correlation and compound correlation

Compound correlation for a tranche is the correlation which when substituted into the one-factor Gaussian copula model produces the market quote for the tranche. Base correlation is the correlation which is consistent with the one-factor Gaussian copula and market quotes for the 0 to X% tranche where X% is a detachment point. It ensures that the expected loss on the 0 to X% tranche equals the sum of the expected losses on the underlying traded tranches.

Problem 23.23.

In the ABS CDO structure in Figure 23.4, suppose that there is a 12% loss on each portfolio. What is the percentage loss experienced by each of the six tranches shown.

The ABS equity tranche is wiped out. There are no losses to the senior ABS tranche. The ABS mezzanine tranche loses $7/20 = 35\%$ of the principal.

Total losses on the ABS CDO are 35\%. The ABS CDO equity and mezzanine tranches are wiped out. The ABS CDO senior tranche loses $10/75 = 13.3\%$ of the principal.

Problem 23.24.

In Example 23.2, what is the tranche spread for the 9% to 12% tranche?

In this case $a_L = 0.09$ and $a_H = 0.12$. Proceeding similarly in Example 23.2 the tranche spread is calculated as 30 basis points.
ASSIGNMENT QUESTIONS

Problem 23.25.

Suppose that the risk-free zero curve is flat at 6% per annum with continuous compounding and that defaults can occur at times 0.25 years, 0.75 years, 1.25 years, and 1.75 years in a two-year plain vanilla credit default swap with semiannual payments. Suppose that the recovery rate is 20% and the unconditional probabilities of default (as seen at time zero) are 1% at times 0.25 years and 0.75 years, and 1.5% at times 1.25 years and 1.75 years. What is the credit default swap spread? What would the credit default spread be if the instrument were a binary credit default swap?

The table corresponding to Table 23.2, giving the present value of the expected regular payments (payment rate is s per year), is

<table>
<thead>
<tr>
<th>Time (years)</th>
<th>Probability of Survival</th>
<th>Expected Payment</th>
<th>Discount Factor</th>
<th>PV of Expected Payment</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.990</td>
<td>0.4950s</td>
<td>0.9704</td>
<td>0.4804s</td>
</tr>
<tr>
<td>1.0</td>
<td>0.980</td>
<td>0.4900s</td>
<td>0.9418</td>
<td>0.4615s</td>
</tr>
<tr>
<td>1.5</td>
<td>0.965</td>
<td>0.4825s</td>
<td>0.9139</td>
<td>0.4410s</td>
</tr>
<tr>
<td>2.0</td>
<td>0.950</td>
<td>0.4750s</td>
<td>0.8869</td>
<td>0.4213s</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td></td>
<td></td>
<td></td>
<td><strong>1.8041s</strong></td>
</tr>
</tbody>
</table>

The table corresponding to Table 23.3, giving the present value of the expected payoffs (notional principal =$1), is

<table>
<thead>
<tr>
<th>Time (years)</th>
<th>Probability of Default</th>
<th>Recovery Rate</th>
<th>Expected Payoff</th>
<th>Discount Factor</th>
<th>PV of Expected Payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>0.010</td>
<td>0.2</td>
<td>0.008</td>
<td>0.9851</td>
<td>0.0079</td>
</tr>
<tr>
<td>0.75</td>
<td>0.010</td>
<td>0.2</td>
<td>0.008</td>
<td>0.9560</td>
<td>0.0076</td>
</tr>
<tr>
<td>1.25</td>
<td>0.015</td>
<td>0.2</td>
<td>0.008</td>
<td>0.9277</td>
<td>0.0111</td>
</tr>
<tr>
<td>1.75</td>
<td>0.015</td>
<td>0.2</td>
<td>0.008</td>
<td>0.9003</td>
<td>0.0108</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td><strong>0.0375</strong></td>
</tr>
</tbody>
</table>

The table corresponding to Table 23.4, giving the present value of accrual payments, is

<table>
<thead>
<tr>
<th>Time (years)</th>
<th>Probability of Default</th>
<th>Expected Accrual Payment</th>
<th>Discount Factor</th>
<th>PV of Expected Accrual Payment</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>0.010</td>
<td>0.0025s</td>
<td>0.9851</td>
<td>0.0025s</td>
</tr>
<tr>
<td>0.75</td>
<td>0.010</td>
<td>0.0025s</td>
<td>0.9560</td>
<td>0.0024s</td>
</tr>
<tr>
<td>1.25</td>
<td>0.015</td>
<td>0.00375s</td>
<td>0.9277</td>
<td>0.0035s</td>
</tr>
<tr>
<td>1.75</td>
<td>0.015</td>
<td>0.00375s</td>
<td>0.9003</td>
<td>0.0034s</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td></td>
<td></td>
<td></td>
<td><strong>0.0117s</strong></td>
</tr>
</tbody>
</table>
The credit default swap spread \( s \) is given by:

\[
1.804s + 0.0117s = 0.0375
\]

It is 0.0206 or 206 basis points. For a binary credit default swap we set the recovery rate equal to zero in the second table to get the present value of expected payoffs equal to 0.0468 so that

\[
1.804s + 0.0117s = 0.0468
\]

and the spread is 0.0258 or 258 basis points.

**Problem 23.26.**

Assume that the default probability for a company in a year, conditional on no earlier defaults is \( \lambda \) and the recovery rate is \( R \). The risk-free interest rate is 5% per annum. Default always occur half way through a year. The spread for a five-year plain vanilla CDS where payments are made annually is 120 basis points and the spread for a five-year binary CDS where payments are made annually is 160 basis points. Estimate \( R \) and \( \lambda \).

The spread for a binary credit default swap is equal to the spread for a regular credit default swap divided by \( 1 - R \) where \( R \) is the recovery rate. This means that \( 1 - R \) equals 0.75 so that the recovery rate is 25%. To find \( \lambda \) we search for the conditional annual default rate that leads to the present value of payments being equal to the present value of payoffs. The answer is \( \lambda = 0.0154 \). The present value of payoffs (per dollar of principal) is then 0.0497. The present value of regular payments is 0.0495. The present value of accrual payments is 0.0002.

**Problem 23.27.**

Explain how you would expect the yields offered on the various tranches in a CDO to change when the correlation between the bonds in the portfolio increases.

As the correlation increases the yield on the equity tranche decreases and the yield on the senior tranches increases. To understand this consider what happens as the correlation increases from zero to one. Initially the equity tranche is mush more risky than the senior tranche. But as the correlation approaches one the companies become essentially the same. We are then in the position where either all companies default or no companies default and the tranches have similar risk.

**Problem 23.28.**

Suppose that

a. The yield on a five-year risk-free bond is 7%

b. The yield on a five-year corporate bond issued by company X is 9.5%

c. A five-year credit default swap providing insurance against company X defaulting costs 150 basis points per year.

What arbitrage opportunity is there in this situation? What arbitrage opportunity would there be if the credit default spread were 300 basis points instead of 150 basis points? Give two reasons why arbitrage opportunities such as those you identify are less than perfect.
When the credit default swap spread is 150 basis points, an arbitrageur can earn more than the risk-free rate by buying the corporate bond and buying protection. If the arbitrageur can finance trades at the risk-free rate (by shorting the riskless bond) it is possible to lock in an almost certain profit of 100 basis points. When the credit spread is 300 basis points the arbitrageur can short the corporate bond, sell protection and buy a risk free bond. This will lock in an almost certain profit of 50 basis points. The arbitrage is not perfect for a number of reasons:

(a) It assumes that both the corporate bond and the riskless bond are par yield bonds and that interest rates are constant. In practice the riskless bond may be worth more or less than par at the time of a default so that a credit default swap underprotects or overprotects the bond holder relative to the position he or she would be in with a riskless bond.

(b) There is uncertainty created by the cheapest-to-delivery bond option.

(c) To be a perfect hedge the credit default swap would have to give the buyer of protection the right to sell the bond for face value plus accrued interest, not just face value.

(d) The arbitrage opportunities assume that market participants can short corporate bonds and borrow at the risk-free rate.

(e) The definition of the credit event in the ISDA agreement is also occasionally a problem. It can occasionally happen that there is a "credit event" but promised payments on the bond are made.

Problem 23.29.

In the ABS CDO structure in Figure 23.4, suppose that there is a 20% loss on each portfolio. What is the percentage loss experienced by each of the six tranches shown.

The ABS equity tranche is wiped out. There are no losses to the senior ABS tranche. The ABS mezzanine tranche loses 15/20 = 75% of the principal. Total losses on the ABS CDO are 75%. The ABS CDO equity and mezzanine tranches are wiped out. The ABS CDO senior tranche loses 50/75 = 66.7% of the principal.

Problem 23.30.

In Example 23.3, what is the spread for a) a first-to-default CDS and b) a second-to-default CDS?

(a) In this case the answer to Example 23.3 gets modified as follows. When \( F = -1.0104 \) the cumulative probabilities of one or more defaults in 1, 2, 3, 4, and 5 years are 0.3103, 0.5435, 0.6997, 0.8027, and 0.8703. The conditional probability that the first default occurs in years 1, 2, 3, 4, and 5 are 0.3103, 0.2332, 0.1562, 0.1030, and 0.0676, respectively. The present values of payoffs, regular payments, and accrual payments conditional on \( F = -1.0104 \) are 0.4784, 1.5900s, and 0.3987s. Similar calculations are carried out for the other factor values. The unconditional expected present values of payoffs, regular payments, and accrual payments are 0.2618, 2.9230s, and 0.2182s. The breakeven spread is therefore

\[
0.2618/(2.9230 + 0.2182) = 0.0833
\]

or 833 basis points. (b) In this case the answer to Example 23.3 gets modified as follows. When \( F = -1.0104 \) the cumulative probabilities of two or more defaults in 1, 2, 3, 4, and
5 years are 0.0493, 0.1711, 0.3159, 0.4551, and 0.5765. The conditional probability that the second default occurs in years 1, 2, 3, 4, and 5 are 0.0493, 0.1219, 0.1447, 0.1392, and 0.1214, respectively. The present values of payoffs, regular payments, and accrual payments conditional on $F = -1.0104$ are 0.3016, 3.0192, and 0.2513. Similar calculations are carried out for the other factor values. The unconditional expected present values of payoffs, regular payments, and accrual payments are 0.1277, 3.7364, and 0.1064. The breakeven spread is therefore

$$0.1277/(3.7364 + 0.1064) = 0.0332$$

or 332 basis points.

**Problem 23.31.**

In Example 23.2, what is the tranche spread for the 6% to 9% tranche?

In this case $a_L = 0.06$ and $a_H = 0.09$. Proceeding similarly in Example 23.2 the tranche spread is calculated as 64 basis points.
CHAPTER 24
Exotic Options

Notes for the Instructor

This chapter now contains material on the variance swaps and volatility swaps, which are becoming increasingly popular. Section 24.13 includes numerical examples to show how they are valued. The calculation of the VIX index is also explained in this section. Another new feature of the chapter is that it distinguishes between fixed and floating lookback options.

This chapter describes a wide range of different exotic options and presents analytic and approximate analytic valuations when these are available. It also discusses static option replication. I do not emphasize the formulas in the chapter. I prefer to spend time talking about how the options work and explaining their properties. Sometimes I use DerivaGem to illustrate the properties of particular options. Problem 24.25, for example, uses DerivaGem to investigate the properties of up-and-out barrier call options.

Problem 24.25 can be a useful part of class discussion when binary options are discussed. Of the assignment questions, 24.27, 24.30, and 24.31 are relatively challenging.

QUESTIONS and PROBLEMS

Problem 24.1.

Explain the difference between a forward start option and a chooser option.

A forward start option is an option that is paid for now but will start at some time in the future. The strike price is usually equal to the price of the asset at the time the option starts. A chooser option is an option where, at some time in the future, the holder chooses whether the option is a call or a put.

Problem 24.2.

Describe the payoff from a portfolio consisting of a lookback call and a lookback put with the same maturity.

A floating lookback call provides a payoff of \( S_T - S_{\text{min}} \). A lookback put provides a payoff of \( S_{\text{max}} - S_T \). A combination of a lookback call and a lookback put therefore provides a payoff of \( S_{\text{max}} - S_{\text{min}} \).

Problem 24.3.

Consider a chooser option where the holder has the right to choose between a European call and a European put at any time during a two-year period. The maturity dates and strike prices for the calls and puts are the same regardless of when the choice is made. Is
it ever optimal to make the choice before the end of the two-year period? Explain your answer.

No, it is never optimal to choose early. The resulting cash flows are the same regardless of when the choice is made. There is no point in the holder making a commitment earlier than necessary. This argument applies when the holder chooses between two American options providing the options cannot be exercised before the 2-year point. If the early exercise period starts as soon as the choice is made, the argument does not hold. For example, if the stock price fell to almost nothing in the first six months, the holder would choose a put option at this time and exercise it immediately.

Problem 24.4.

Suppose that $c_1$ and $p_1$ are the prices of a European average price call and a European average price put with strike price $K$ and maturity $T$, $c_2$ and $p_2$ are the prices of a European average strike call and European average strike put with maturity $T$, and $c_3$ and $p_3$ are the prices of a regular European call and a regular European put with strike price $K$ and maturity $T$. Show that

$$c_1 + c_2 - c_3 = p_1 + p_2 - p_3$$

The payoffs are as follows:

- $c_1 : \max(S - K, 0)$
- $c_2 : \max(S_T - S, 0)$
- $c_3 : \max(S_T - K, 0)$
- $p_1 : \max(K - S, 0)$
- $p_2 : \max(S - S_T, 0)$
- $p_3 : \max(K - S_T, 0)$

The payoff from $c_1 - p_1$ is always $S - K$; The payoff from $c_2 - p_2$ is always $S_T - S$; The payoff from $c_3 - p_3$ is always $S_T - K$; It follows that

$$c_1 - p_1 + c_2 - p_2 = c_3 - p_3$$

or

$$c_1 + c_2 - c_3 = p_1 + p_2 - p_3$$

Problem 24.5.

The text derives a decomposition of a particular type of chooser option into a call maturing at time $T_2$ and a put maturing at time $T_1$. Derive an alternative decomposition into a call maturing at time $T_1$ and a put maturing at time $T_2$.

Substituting for $c$, put-call parity gives

$$\max(c, p) = \max \left[ p, p + S_1 e^{-q(T_2 - T_1)} - Ke^{-r(T_2 - T_1)} \right]$$

$$= p + \max \left[ 0, S_1 e^{-q(T_2 - T_1)} - Ke^{-r(T_2 - T_1)} \right]$$

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This shows that the chooser option can be decomposed into
1. A put option with strike price $K$ and maturity $T_2$; and
2. $e^{-q(T_2-T_1)}$ call options with strike price $Ke^{-(r-q)(T_2-T_1)}$ and maturity $T_1$.

**Problem 24.6.**

Section 24.6 gives two formulas for a down-and-out call. The first applies to the situation where the barrier, $H$, is less than or equal to the strike price, $K$. The second applies to the situation where $H \geq K$. Show that the two formulas are the same when $H = K$.

Consider the formula for $C_{do}$ when $H \geq K$

$$c_{do} = S_0N(x_1)e^{-qT} - Ke^{-rT}N(x_1 - \sigma \sqrt{T}) - S_0e^{-qT}(H/S_0)^{2\lambda}N(y_1) + Ke^{-rT}(H/S_0)^{2\lambda-2}N(y_1 - \sigma \sqrt{T})$$

Substituting $H = K$ and noting that

$$\lambda = \frac{r - q + \sigma^2/2}{\sigma^2}$$

we obtain $x_1 = d_1$ so that

$$c_{do} = c - S_0e^{-qT}(H/S_0)^{2\lambda}N(y_1) + Ke^{-rT}(H/S_0)^{2\lambda-2}N(y_1 - \sigma \sqrt{T})$$

The formula for $c_{di}$ when $H \leq K$ is

$$c_{di} = S_0e^{-qT}(H/S_0)^{2\lambda}N(y) - Ke^{-rT}(H/S_0)^{2\lambda-2}N(y - \sigma \sqrt{T})$$

Since $c_{do} = c - c_{di}$

$$c_{do} = c - S_0e^{-qT}(H/S_0)^{2\lambda}N(y) + Ke^{-rT}(H/S_0)^{2\lambda-2}N(y - \sigma \sqrt{T})$$

From the formulas in the text $y_1 = y$ when $H = K$. The two expression for $c_{do}$ are therefore equivalent when $H = K$.

**Problem 24.7.**

Explain why a down-and-out put is worth zero when the barrier is greater than the strike price.

The option is in the money only when the asset price is less than the strike price. However, in these circumstances the barrier has been hit and the option has ceased to exist.
Problem 24.8.

Suppose that the strike price of an American call option on a non-dividend-paying stock grows at rate \( g \). Show that if \( g \) is less than the risk-free rate, \( r \), it is never optimal to exercise the call early.

The argument is similar to that given in Chapter 9 for a regular option on a non-dividend-paying stock. Consider a portfolio consisting of the option and cash equal to the present value of the terminal strike price. The initial cash position is

\[
Ke^{gT-rT}
\]

By time \( \tau \) (\( 0 \leq \tau \leq T \)), the cash grows to

\[
Ke^{\nu(T-\tau)+gT} = Ke^{gT}e^{-(r-g)(T-\tau)}
\]

Since \( r > g \), this is less than \( Ke^{gT} \) and therefore is less than the amount required to exercise the option. It follows that, if the option is exercised early, the terminal value of the portfolio is less than \( ST \). At time \( T \) the cash balance is \( Ke^{gT} \). This is exactly what is required to exercise the option. If the early exercise decision is delayed until time \( T \), the terminal value of the portfolio is therefore

\[
\max[ST, Ke^{gT}]
\]

This is at least as great as \( ST \). It follows that early exercise cannot be optimal.

Problem 24.9.

How can the value of a forward start put option on a non-dividend-paying stock be calculated if it is agreed that the strike price will be 10% greater than the stock price at the time the option starts?

When the strike price of an option on a non-dividend-paying stock is defined as 10% greater than the stock price, the value of the option is proportional to the stock price. The same argument as that given in the text for forward start options shows that if \( t_1 \) is the time when the option starts and \( t_2 \) is the time when it finishes, the option has the same value as an option starting today with a life of \( t_2 - t_1 \) and a strike price of 1.1 times the current stock price.

Problem 24.10.

If a stock price follows geometric Brownian motion, what process does \( A(t) \) follow where \( A(t) \) is the arithmetic average stock price between time zero and time \( t \)?

Assume that we start calculating averages from time zero. The relationship between \( A(t + \Delta t) \) and \( A(t) \) is

\[
A(t + \Delta t) \times (t + \Delta t) = A(t) \times t + S(t) \times \Delta t
\]

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where \( S(t) \) is the stock price at time \( t \) and terms of higher order than \( \Delta t \) are ignored. If we continue to ignore terms of higher order than \( \Delta t \), it follows that

\[
A(t + \Delta t) = A(t) \left[ 1 - \frac{\Delta t}{t} \right] + S(t) \frac{\Delta t}{t}
\]

Taking limits as \( \Delta t \) tends to zero

\[
dA(t) = \frac{S(t) - A(t)}{t} dt
\]

The process for \( A(t) \) has a stochastic drift and no \( dz \) term. The process makes sense intuitively. Once some time has passed, the change in \( S \) in the next small portion of time has only a second order effect on the average. If \( S \) equals \( A \) the average has no drift; if \( S > A \) the average is drifting up; if \( S < A \) the average is drifting down.

**Problem 24.11.**

*Explain why delta hedging is easier for Asian options than for regular options.*

In an Asian option the payoff becomes more certain as time passes and the delta always approaches zero as the maturity date is approached. This makes delta hedging easy. Barrier options cause problems for delta hedgers when the asset price is close to the barrier because delta is discontinuous.

**Problem 24.12.**

*Calculate the price of a one-year European option to give up 100 ounces of silver in exchange for one ounce of gold. The current prices of gold and silver are \$380 and \$4, respectively; the risk-free interest rate is 10% per annum; the volatility of each commodity price is 20%; and the correlation between the two prices is 0.7. Ignore storage costs.*

The value of the option is given by the formula in the text

\[
V_0 e^{-q_2 T} N(d_1) - U_0 e^{-q_1 T} N(d_2)
\]

where

\[
d_1 = \frac{\ln(V_0 / U_0) + (q_1 - q_2 + \sigma^2 / 2)T}{\sigma \sqrt{T}}
\]

\[
d_2 = d_1 - \sigma \sqrt{T}
\]

and

\[
\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2 \rho \sigma_1 \sigma_2}
\]

In this case, \( V_0 = 380, U_0 = 400, q_1 = 0, q_2 = 0, T = 1, \) and

\[
\sigma = \sqrt{0.2^2 + 0.2^2 - 2 \times 0.7 \times 0.2 \times 0.2} = 0.1549
\]
Because $d_1 = -0.2537$ and $d_2 = -0.4086$, the option price is

$$380N(-0.2537) - 400N(-0.4086) = 15.38$$

or $\$15.38$.

**Problem 24.13.**

Is a European down-and-out option on an asset worth the same as a European down-and-out option on the asset’s futures price for a futures contract maturing at the same time as the option?

No. If the future’s price is above the spot price during the life of the option, it is possible that the spot price will hit the barrier when the futures price does not.

**Problem 24.14.**

Answer the following questions about compound options

(a) What put–call parity relationship exists between the price of a European call on a call and a European put on a call? Show that the formulas given in the text satisfy the relationship.

(b) What put–call parity relationship exists between the price of a European call on a put and a European put on a put? Show that the formulas given in the text satisfy the relationship.

(a) The put–call relationship is

$$cc + K_1 e^{-rT_1} = pc + c$$

where $cc$ is the price of the call on the call, $pc$ is the price of the put on the call, $c$ is the price today of the call into which the options can be exercised at time $T_1$, and $K_1$ is the exercise price for $cc$ and $pc$. The proof is similar to that in Chapter 9 for the usual put–call parity relationship. Both sides of the equation represent the values of portfolios that will be worth $\max(c, K_1)$ at time $T_1$. Because

$$M(a, b; \rho) = N(a) - M(a, -b; -\rho) = N(b) - M(-a, b; -\rho)$$

and

$$N(x) = 1 - N(-x)$$

we obtain

$$cc - pc = Se^{-qT_2}N(b_1) - K_2 e^{-rT_2}N(b_2) - K_1 e^{-rT_1}$$

Since

$$c = Se^{-qT_2}N(b_1) - K_2 e^{-rT_2}N(b_2)$$

put–call parity is consistent with the formulas

(b) The put–call relationship is

$$cp + K_1 e^{-rT_1} = pp + p$$
where \( cp \) is the price of the call on the put, \( pp \) is the price of the put on the put, \( p \) is the price today of the put into which the options can be exercised at time \( T_1 \), and \( K_1 \) is the exercise price for \( cp \) and \( pp \). The proof is similar to that in Chapter 9 for the usual put-call parity relationship. Both sides of the equation represent the values of portfolios that will be worth \( \max(p, K_1) \) at time \( T_1 \). Because

\[
M(a, b; \rho) = N(a) - M(a, -b; -\rho) = N(b) - M(-a, b; -\rho)
\]

and

\[
N(x) = 1 - N(-x)
\]

it follows that

\[
cp - pp = -Se^{-qT_2}N(-b_1) + K_2e^{-rT_2}N(-b_2) - K_1e^{-rT_1}
\]

Because

\[
p = -Se^{-qT_2}N(-b_1) + K_2e^{-rT_2}N(-b_2)
\]

put-call parity is consistent with the formulas.

**Problem 24.15.**

*Does a floating lookback call become more valuable or less valuable as we increase the frequency with which we observe the asset price in calculating the minimum?*

As we increase the frequency we observe a more extreme minimum which increases the value of a lookback call.

**Problem 24.16.**

*Does a down-and-out call become more valuable or less valuable as we increase the frequency with which we observe the asset price in determining whether the barrier has been crossed? What is the answer to the same question for a down-and-in call?*

As we increase the frequency with which the asset price is observed, the asset price becomes more likely to hit the barrier and the value of a down-and-out call goes down. For a similar reason the value of a down-and-in call goes up. The adjustment mentioned in the text, suggested by Broadie, Glasserman, and Kou, moves the barrier further out as the asset price is observed less frequently. This increases the price of a down-and-out option and reduces the price of a down-and-in option.

**Problem 24.17.**

*Explain why a regular European call option is the sum of a down-and-out European call and a down-and-in European call. Is the same true for American call options?*

If the barrier is reached the down-and-out option is worth nothing while the down-and-in option has the same value as a regular option. If the barrier is not reached the down-and-in option is worth nothing while the down-and-out option has the same value as a regular option. This is why a down-and-out call option plus a down-and-in call option
is worth the same as a regular option. A similar argument cannot be used for American options.

**Problem 24.18.**
What is the value of a derivative that pays off $100 in six months if the S&P 500 index is greater than 1,000 and zero otherwise? Assume that the current level of the index is 960, the risk-free rate is 8% per annum, the dividend yield on the index is 3% per annum, and the volatility of the index is 20%.

This is a cash-or-nothing call. The value is $100 N(d_2) e^{-0.08 	imes 0.5}$ where

$$d_2 = \frac{\ln(960/1000) + (0.08 - 0.03 - 0.2^2/2) 	imes 0.5}{0.2 \times \sqrt{0.5}} = -0.1826$$

Since $N(d_2) = 0.4276$ the value of the derivative is $41.08.

**Problem 24.19.**
In a three-month down-and-out call option on silver futures the strike price is $20 per ounce and the barrier is $18. The current futures price is $19, the risk-free interest rate is 5%, and the volatility of silver futures is 40% per annum. Explain how the option works and calculate its value. What is the value of a regular call option on silver futures with the same terms? What is the value of a down-and-in call option on silver futures with the same terms?

This is a regular call with a strike price of $20 that ceases to exist if the futures price hits $18. With the notation in the text $H = 18$, $K = 20$, $S = 19$, $r = 0.05$, $\sigma = 0.4$, $q = 0.05$, $T = 0.25$. From this $\lambda = 0.5$ and

$$y = \frac{\ln[18^2/(19 \times 20)]}{0.4 \sqrt{0.25}} + 0.5 \times 0.4 \sqrt{0.25} = -0.69714$$

The value of a down-and-out call plus a down-and-in call equals the value of a regular call. Substituting into the formula given when $H < K$ we get $c_{di} = 0.4638$. The regular Black–Scholes formula gives $c = 1.0902$. Hence $c_{do} = 0.6264$. (These answers can be checked with DerivaGem.)

**Problem 24.20.**
A new European-style floating lookback call option on a stock index has a maturity of nine months. The current level of the index is 400, the risk-free rate is 6% per annum, the dividend yield on the index is 4% per annum, and the volatility of the index is 20%. Use DerivaGem to value the option.

DerivaGem shows that the value is 53.38. Note that the Minimum to date and Maximum to date should be set equal to the current value of the index for a new deal. (See material on DerivaGem at the end of the book.)
Problem 24.21.

Estimate the value of a new six-month European-style average price call option on a non-dividend-paying stock. The initial stock price is $30, the strike price is $30, the risk-free interest rate is 5%, and the stock price volatility is 30%.

We can use the analytic approximation given in the text.

\[
M_1 = \frac{(e^{0.05 \times 0.5} - 1) \times 30}{0.05 \times 0.5} = 30.378
\]

Also \( M_2 = 936.9 \) so that \( \sigma = 17.41\% \). The option can be valued as a futures option with \( F_0 = 30.378, K = 30, r = 5\%, \sigma = 17.41\%, \) and \( t = 0.5 \). The price is 1.637.

Problem 24.22.

Use DerivaGem to calculate the value of:

(a) A regular European call option on a non-dividend-paying stock where the stock price is $50, the strike price is $50, the risk-free rate is 5% per annum, the volatility is 30%, and the time to maturity is one year.

(b) A down-and-out European call which is as in (a) with the barrier at $45.

(c) A down-and-in European call which is as in (a) with the barrier at $45.

Show that the option in (a) is worth the sum of the values of the options in (b) and (c).

(a) The price of a regular European call option is 7.116.

(b) The price of the down-and-out call option is 4.696.

(c) The price of the down-and-in call option is 2.419.

The price of a regular European call is the sum of the prices of down-and-out and down-and-in options.

Problem 24.23.

Explain adjustments that have to be made when \( r = q \) for a) the valuation formulas for lookback call options in Section 24.8 and b) the formulas for \( M_1 \) and \( M_2 \) in Section 24.10.

When \( r = q \) in the expression for a lookback call in Section 24.8 \( a_1 = a_3 \) and \( Y_1 = \ln(S_0/S_{\min}) \) so that the expression for a lookback call becomes

\[
S_0e^{-qT}N(a_1) - S_{\min}e^{-rT}N(a_2)
\]

As \( q \) approaches \( r \) in Section 24.10 we get

\[
M_1 = S_0
\]

\[
M_2 = \frac{2e^{\sigma^2T_S^2}}{\sigma^4T^2} - \frac{2S_0^2}{T^2} 1 + \frac{\sigma^2T}{\sigma^4}
\]
Problem 24.24.

Value the variance swap in Example 24.3 of Section 24.13 assuming that the implied volatilities for options with strike prices 800, 850, 900, 950, 1,000, 1,050, 1,100, 1,150, 1,200 are 20%, 20.5%, 21%, 21.5%, 22%, 22.5%, 23%, 23.5%, 24%, respectively.

In this case, DerivaGem shows that $Q(K_1) = 0.1772$, $Q(K_2) = 1.1857$, $Q(K_3) = 4.9123$, $Q(K_4) = 14.2374$, $Q(K_5) = 45.3738$, $Q(K_6) = 35.9243$, $Q(K_7) = 20.6883$, $Q(K_8) = 11.4135$, $Q(K_9) = 6.1043$. $\hat{E}(\overline{V}) = 0.0502$. The value of the variance swap is $0.51$ million.
ASSIGNMENT QUESTIONS

Problem 24.25.

What is the value in dollars of a derivative that pays off £10,000 in one year provided that the dollar-sterling exchange rate is greater than 1.5000 at that time? The current exchange rate is 1.4800. The dollar and sterling interest rates are 4% and 8% per annum respectively. The volatility of the exchange rate is 12% per annum.

It is instructive to consider two different ways of valuing this instrument. From the perspective of a sterling investor it is a cash or nothing put. The variables are $S_0 = 1/1.48 = 0.6757$, $K = 1/1.50 = 0.6667$, $r = 0.08$, $q = 0.04$, $\sigma = 0.12$, and $T = 1$. The derivative pays off if the exchange rate is less than 0.6667. The value of the derivative is $10,000N(-d_2)e^{-0.08\times1}$ where

$$d_2 = \frac{\ln(0.6757/0.6667) + (0.08 - 0.04 - 0.12^2/2)}{0.12} = 0.3852$$

Since $N(-d_2) = 0.3501$, the value of the derivative is $10,000 \times 0.3501 \times e^{-0.08} = 3,231$ or $3,231$. In dollars this is $3,231 \times 1.48 = $4782.

From the perspective of a dollar investor the derivative is an asset or nothing call. The variables are $S_0 = 1.48$, $K = 1.50$, $r = 0.04$, $q = 0.08$, $\sigma = 0.12$ and $T = 1$. The value is $10,000N(d_1)e^{-0.08\times1}$ where

$$d_1 = \frac{\ln(1.48/1.50) + (0.04 - 0.08 + 0.12^2/2)}{0.12} = -0.3852$$

$N(d_1) = 0.3500$ and the value of the derivative is as before $10,000 \times 1.48 \times 0.3500 \times e^{-0.08} = 4,782$ or $4,782$.


Consider an up-and-out barrier call option on a non-dividend-paying stock when the stock price is 50, the strike price is 50, the volatility is 30%, the risk-free rate is 5%, the time to maturity is one year, and the barrier at $80. Use the software to value the option and graph the relationship between (a) the option price and the stock price, (b) the delta and the option price, (c) the option price and the time to maturity, and (d) the option price and the volatility. Provide an intuitive explanation for the results you get. Show that the delta, gamma, theta, and vega for an up-and-out barrier call option can be either positive or negative.

The price of the option is 3.528.

(a) The option price is a humped function of the stock price with the maximum option price occurring for a stock price of about $57. If you could choose the stock price there would be a trade off. High stock prices give a high probability that the option will be knocked out. Low stock prices give a low potential payoff. For a stock price less than $57$ delta is positive (as it is for a regular call option); for a stock price greater than $57$, delta is negative.
(b) Delta increases up to a stock price of about 45 and then decreases. This shows that gamma can be positive or negative.

(c) The option price is a humped function of the time to maturity with the maximum option price occurring for a time to maturity of 0.5 years. This is because too long a time to maturity means that the option has a high probability of being knocked out; too short a time to maturity means that the option has a low potential payoff. For a time to maturity less than 0.5 years theta is negative (as it is for a regular call option); for a time to maturity greater than 0.5 years the theta of the option is positive.

(d) The option price is also a humped function of volatility with the maximum option price being obtained for a volatility of about 20%. This is because too high a volatility means that the option has a high probability of being knocked out; too low a volatility means that the option has a low potential payoff. For volatilities less than 20% vega is positive (as it is for a regular option); for volatilities above 20% vega is negative.

Problem 24.27.

Sample Application F in the DerivaGem Application Builder Software considers the static options replication example in Section 24.13. It shows the way a hedge can be constructed using four options (as in Section 24.13) and two ways a hedge can be constructed using 16 options.

a. Explain the difference between the two ways a hedge can be constructed using 16 options. Explain intuitively why the second method works better.

b. Improve on the four-option hedge by changing Tmat for the third and fourth options.

c. Check how well the 16-option portfolios match the delta, gamma, and vega of the barrier option.

(a) Both approaches use a one call option with a strike price of 50 and a maturity of 0.75. In the first approach the other 15 call options have strike prices of 60 and equally spaced times to maturity. In the second approach the other 15 call options have strike prices of 60, but the spacing between the times to maturity decreases as the maturity of the barrier option is approached. The second approach to setting times to maturity produces a better hedge. This is because the chance of the barrier being hit at time $t$ is an increasing function of $t$. As $t$ increases it therefore becomes more important to replicate the barrier at time $t$.

(b) By using either trial and error or the Solver tool we see that we come closest to matching the price of the barrier option when the maturities of the third and fourth options are changed from 0.25 and 0.5 to 0.39 and 0.65.

(c) To calculate delta for the two 16-option hedge strategies it is necessary to change the last argument of EPortfolio from 0 to 1 in cells L42 and X42. To calculate delta for the barrier option it is necessary to change the last argument of BarrierOption in cell F12 from 0 to 1. To calculate gamma and vega the arguments must be changed to 2 and 3, respectively. The delta, gamma, and vega of the barrier option are $-0.0221$, $-0.0035$, and $-0.0254$. The delta, gamma, and vega of the first 16-option portfolio are $-0.0262$, $-0.0045$, and $-0.1470$. The delta, gamma, and vega of the second 16-option portfolio are $-0.0199$, $-0.0037$, and $-0.1449$. The second of the two 16-option portfolios provides
Greek letters that are closest to the Greek letters of the barrier option. Interestingly neither of the two portfolios does particularly well on vega.

**Problem 24.28**

Consider a down-and-out call option on a foreign currency. The initial exchange rate is 0.90, the time to maturity is two years, the strike price is 1.00, the barrier is 0.80, the domestic risk-free interest rate is 5%, the foreign risk-free interest rate is 6%, and the volatility is 25% per annum. Use DerivaGem to develop a static option replication strategy involving five options.

A natural approach is to attempt to replicate the option with positions in:

(a) A European call option with strike price 1.00 maturing in two years
(b) A European put option with strike price 0.80 maturing in two years
(c) A European put option with strike price 0.80 maturing in 1.5 years
(d) A European put option with strike price 0.80 maturing in 1.0 years
(e) A European put option with strike price 0.80 maturing in 0.5 years

The first option can be used to match the value of the down-and-out-call for $t = 2$ and $S > 1.00$. The others can be used to match it at the following $(t, S)$ points: $(1.5, 0.80)$, $(1.0, 0.80)$, $(0.5, 0.80)$, $(0.0, 0.80)$. Following the procedure in the text, we find that the required positions in the options are as shown in the following table.

<table>
<thead>
<tr>
<th>Option Type</th>
<th>Strike Price</th>
<th>Maturity (years)</th>
<th>Position</th>
</tr>
</thead>
<tbody>
<tr>
<td>Call</td>
<td>1.00</td>
<td>2.00</td>
<td>+1.0000</td>
</tr>
<tr>
<td>Put</td>
<td>0.80</td>
<td>2.00</td>
<td>-0.1255</td>
</tr>
<tr>
<td>Put</td>
<td>0.80</td>
<td>1.50</td>
<td>-0.1758</td>
</tr>
<tr>
<td>Put</td>
<td>0.80</td>
<td>1.00</td>
<td>-0.0956</td>
</tr>
<tr>
<td>Put</td>
<td>0.80</td>
<td>0.50</td>
<td>-0.0547</td>
</tr>
</tbody>
</table>

The value of the portfolio initially is 0.482. This is only a little less than the value of the down-and-out-option which is 0.488. This example is different from the example in the text in a number of ways. Put options and call options are used in the replicating portfolio. The value of the replicating portfolio converges to the value of the option from below rather than from above. Also, even with relatively few options, the value of the replicating portfolio is close to the value of the down-and-out option.

**Problem 24.29.**

Suppose that a stock index is currently 900. The dividend yield is 2%, the risk-free rate is 5%, and the volatility is 40%. Use the results in the appendix to calculate the value of a one-year average price call where the strike price is 900 and the index level is observed at the end of each quarter for the purposes of the averaging. Compare this with the price calculated by DerivaGem for a one-year average price option where the price is observed continuously. Provide an intuitive explanation for any differences between the prices.

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In this case $M_1 = 917.07$ and $M_2 = 904028.7$ so that the option can be valued as an option on futures where the futures price is 917.07 and volatility is $\sqrt{\ln(904028.7/917.07^2)}$ or 26.88%. The value of the option is 100.74. DerivaGem gives the price as 86.77 (set option type = Asian). The higher price for the first option arises because the average is calculated from prices at times 0.25, 0.50, 0.75, and 1.00. The mean of these times is 0.625. By contrast the corresponding mean when the price is observed continuously is 0.50. The later a price is observed the more uncertain it is and the more it contributes to the value of the option.

**Problem 24.30.**

*Use the DerivaGem Application Builder software to compare the effectiveness of daily delta hedging for (a) the option considered in Tables 17.2 and 17.3 and (b) an average price call with the same parameters. Use Sample Application C. For the average price option you will find it necessary to change the calculation of the option price in cell C16, the payoffs in cells H15 and H16, and the deltas (cells G46 to G186 and N46 to N186). Carry out 20 Monte Carlo simulation runs for each option by repeatedly pressing F9. On each run record the cost of writing and hedging the option, the volume of trading over the whole 20 weeks and the volume of trading between weeks 11 and 20. Comment on the results.*

For the regular option the theoretical price is 239,599. For the average price option the theoretical price is 115,259. My 20 simulation runs (40 outcomes because of the antithetic calculations) gave results as shown in the following table.

<table>
<thead>
<tr>
<th></th>
<th>Regular Call</th>
<th>Ave Price Call</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ave. Hedging Cost</td>
<td>247,628</td>
<td>114,837</td>
</tr>
<tr>
<td>S.D. Hedging Cost</td>
<td>17,833</td>
<td>12,123</td>
</tr>
<tr>
<td>Ave Trading Vol (20 wks)</td>
<td>412,440</td>
<td>291,237</td>
</tr>
<tr>
<td>Ave Trading Vol (last 10 wks)</td>
<td>187,074</td>
<td>51,658</td>
</tr>
</tbody>
</table>

These results show that the standard deviation of using delta hedging for an average price option is lower than that for a regular option. However, using the criterion in Chapter 17 (standard deviation divided by value of option) hedge performance is better for the regular option. Hedging the average price option requires less trading, particularly in the last 10 weeks. This is because we become progressively more certain about the average price as the maturity of the option is approached.

**Problem 24.31**

*In the DerivaGem Application Builder Software modify Sample Application D to test the effectiveness of delta and gamma hedging for a call on call compound option on a 100,000 units of a foreign currency where the exchange rate is 0.67, the domestic risk-free rate is 5%, the foreign risk-free rate is 6%, the volatility is 12%. The time to maturity of*
the first option is 20 weeks, and the strike price of the first option is 0.015. The second option matures 40 weeks from today and has a strike price of 0.68. Explain how you modified the cells. Comment on hedge effectiveness.

The value of the option is 1093. It is necessary to change cells F20 and F46 to 0.67. Cells G20 to G39 and G46 to G65 must be changed to calculate delta of the compound option. Cells H20 to H39 and H46 to H65 must be changed to calculate gamma of the compound option. Cells I20 to I40 and I46 to I66 must be changed to calculate the Black–Scholes price of the call option expiring in 40 weeks. Similarly cells J20 to J40 and J46 to J66 must be changed to calculate the delta of this option; cells K20 to K40 and K46 to K66 must be changed to calculate the gamma of the option. The payoffs in cells N9 and N10 must be calculated as MAX(140-0.015, 0)*100000 and MAX(166-0.015, 0)*100000. Delta plus gamma hedging works relatively poorly for the compound option. On 20 simulation runs the cost of writing and hedging the option ranged from 200 to 2500.

Problem 24.32.

Outperformance certificates (also called “sprint certificates”, “accelerator certificates”, or “speeders”) are offered to investors by many European banks as a way of investing in a company’s stock. The initial investment equals the company’s stock price, $S_0$. If the stock price goes up between time 0 and time $T$, the investor gains $k$ times the increase at time $T$ where $k$ is a constant greater than 1.0. However, the stock price used to calculate the gain at time $T$ is capped at some maximum level $M$. If the stock price goes down the investor’s loss is equal to the decrease. The investor does not receive dividends.

a) Show that the net gain from an outperformance certificate is a package.

b) Calculate using DerivaGem the value of a one-year outperformance certificate when the stock price is 50 euros, $k = 1.5$, $M = 70$ euros, the risk-free rate is 5%, and the stock price volatility is 25%. Dividends equal to 0.5 euros are expected in 2 months, 5 month, 8 months, and 11 months.

a) The investor’s gain (loss) on an initial investment of $S_0$ is equivalent to:
1. A long position in $k$ one-year European call options on the stock with a strike price equal to the current stock price.
2. A short position in $k$ one-year European call options on the stock with a strike price equal to $M$
3. A short position in one European one-year put option on the stock with a strike price equal to the current stock price.

b) In this case the value of the three parts to the gain are
1. $1.5 \times 5.0056 = 7.5084$
2. $-1.5 \times 0.6339 = 0.9509$
3. $-4.5138$

The total value of the gain is $7.5084 - 0.9509 - 4.5138 = 2.0437$

Problem 24.33.

Carry out the analysis in Example 24.3 of Section 24.13 to value the variance swap on the assumption that the life of the swap is 1 month rather than 3 months.
In this case, \( F_0 = 1022.55 \) and DerivaGem shows that \( Q(K_1) = 0.0366 \), \( Q(K_2) = 0.2858 \), \( Q(K_3) = 1.5822 \), \( Q(K_4) = 6.3708 \), \( Q(K_5) = 30.3864 \), \( Q(K_6) = 16.9233 \), \( Q(K_7) = 4.8180 \), \( Q(K_8) = 0.8639 \), and \( Q_9 = 0.0863 \). \( \mathbb{E}(\mathcal{V}) = 0.0661 \). The value of the variance swap is $2.09 million.
CHAPTER 25
Weather, Energy, and Insurance Derivatives

Notes for the Instructor

The chapter describes a number of nontraditional derivatives products. The market for weather derivatives is relatively new and quite small. By contrast, energy derivatives (especially oil derivatives) have been around for a long time. It is interesting to talk about the different ways in which oil, gas, and electricity derivatives products are structured to accommodate the different properties of these energy sources. The section on insurance derivatives provides a thumbnail sketch of how the reinsurance industry works and explains some of the derivative products that have been developed as alternates to traditional forms of reinsurance.

Students enjoy the material in this chapter and find the descriptions of how the different products work fascinating. I like to distinguish between the historical data approach to valuation and the risk-neutral valuation approach. (See the first section of the chapter.) When underlying market variables have no systematic risk (as they often do in the derivatives considered in this chapter) the historical data and risk-neutral valuation approaches are equivalent.

Students enjoy the material in this chapter and find the descriptions of how the different products work fascinating. Problem 25.15 can be used as an assignment question or for class discussion.

QUESTIONS AND PROBLEMS

Problem 25.1.
What is meant by HDD and CDD?

A day's HDD is max(0, 65 - A) and a day's CDD is max(0, A - 65) where \( A \) is the average of the highest and lowest temperature during the day at a specified weather station, measured in degrees Fahrenheit.

Problem 25.2.
How is a typical natural gas forward contract structured?

It is an agreement by one side to delivery a specified amount of gas at a roughly uniform rate during a month to a particular hub for a specified price.

Problem 25.3.
Distinguish between the historical data and the risk-neutral approach to valuing a derivative. Under what circumstance do they give the same answer.
The historical data approach to valuing an option involves calculating the expected payoff using historical data and discounting the payoff at the risk-free rate. The risk-neutral approach involves calculating the expected payoff in a risk-neutral world and discounting at the risk-free rate. The two approaches give the same answer when percentage changes in the underlying market variables have zero correlation with stock market returns. (In these circumstances all risks can be diversified away.)

**Problem 25.4.**

Suppose that each day during July the minimum temperature is 68° Fahrenheit and the maximum temperature is 82° Fahrenheit. What is the payoff from a call option on the cumulative CDD during July with a strike of 250 and a payment rate of $5,000 per degree day?

The average temperature each day is 75°. The CDD each day is therefore 10 and the cumulative CDD for the month is $10 \times 31 = 310$. The payoff from the call option is therefore $(310 - 250) \times 5,000 = $300,000.

**Problem 25.5.**

*Why is the price of electricity more volatile than that of other energy sources?*

Unlike most commodities electricity cannot be stored easily. If the demand for electricity exceeds the supply, as it sometimes does during the air conditioning season, the price of electricity in a deregulated environment will skyrocket. When supply and demand become matched again the price will return to former levels.

**Problem 25.6.**

*Why is the historical data approach appropriate for pricing a weather derivatives contract and a CAT bond?*

There is no systematic risk (i.e., risk that is priced by the market) in weather derivatives and CAT bonds.

**Problem 25.7.**

"HDD and CDD can be regarded as payoffs from options on temperature." Explain

HDD is $\max(65 - A, 0)$ where $A$ is the average of the maximum and minimum temperature during the day. This is the payoff from a put option on $A$ with a strike price of 65. CDD is $\max(A - 65, 0)$. This is the payoff from call option on $A$ with a strike price of 65.

**Problem 25.8.**

Suppose that you have 50 years of temperature data at your disposal. Explain carefully the analyses you would carry out to value a forward contract on the cumulative CDD for a particular month.

It would be useful to calculate the cumulative CDD for July of each year of the last 50 years. A linear regression relationship

$$CDD = a + bt + e$$

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could then be estimated where \( a \) and \( b \) are constants, \( t \) is the time in years measured from the start of the 50 years, and \( e \) is the error. This relationship allows for linear trends in temperature through time. The expected CDD for next year (year 51) is then \( a + 51b \). This could be used as an estimate of the forward CDD.

**Problem 25.9.**

Would you expect the volatility of the one-year forward price of oil to be greater than or less than the volatility of the spot price. Explain.

The volatility of the three-month forward price will be less than the volatility of the spot price. This is because, when the spot price changes by a certain amount, mean reversion will cause the forward price will change by a lesser amount.

**Problem 25.10.**

What are the characteristics of an energy source where the price has a very high volatility and a very high rate of mean reversion? Give an example of such an energy source.

The price of the energy source will show big changes, but will be pulled back to its long-run average level fast. Electricity is an example of an energy source with these characteristics.

**Problem 25.11.**

How can an energy producer use derivative markets to hedge risks?

The energy producer faces quantity risks and price risks. It can use weather derivatives to hedge the quantity risks and energy derivatives to hedge against the price risks.

**Problem 25.12.**

Explain how a 5 x 8 option contract for May 2009 on electricity with daily exercise works. Explain how a 5 x 8 option contract for May 2009 on electricity with monthly exercise works. Which is worth more?

A 5 x 8 contract for May, 2009 is a contract to provide electricity for five days per week during the off-peak period (11PM to 7AM). When daily exercise is specified, the holder of the option is able to choose each weekday whether he or she will buy electricity at the strike price at the agreed rate. When there is monthly exercise, he or she chooses once at the beginning of the month whether electricity is to be bought at the strike price at the agreed rate for the whole month. The option with daily exercise is worth more.

**Problem 25.13.**

Explain how CAT bonds work.

CAT bonds (catastrophe bonds) are an alternative to reinsurance for an insurance company that has taken on a certain catastrophic risk (e.g., the risk of a hurricane or an earthquake) and wants to get rid of it. CAT bonds are issued by the insurance company. They provide a higher rate of interest than government bonds. However, the bondholders
agree to forego interest, and possibly principal, to meet any claims against the insurance company that are within a prespecified range.

**Problem 25.14.**

Consider two bonds that have the same coupon, time to maturity and price. One is a B-rated corporate bond. The other is a CAT bond. An analysis based on historical data shows that the expected losses on the two bonds in each year of their life is the same. Which bond would you advise a portfolio manager to buy and why?

The CAT bond has very little systematic risk. Whether a particular type of catastrophe occurs is independent of the return on the market. The risks in the CAT bond are likely to be largely “diversified away” by the other investments in the portfolio. A B-rated bond does have systematic risk that cannot be diversified away. It is likely therefore that the CAT bond is a better addition to the portfolio.

**ASSIGNMENT QUESTIONS**

**Problem 25.15.**

(a) The losses in millions of dollars are approximately

\[ \phi(150, 50^2) \]

The reinsurance contract would pay out 60% of the losses. The payout from the reinsurance contract is therefore

\[ \phi(90, 30^2) \]

The cost of the reinsurance is the expected payout in a risk-neutral world, discounted at the risk-free rate. In this case, the expected payout is the same in a risk-neutral world as it is in the real world. The value of the reinsurance contract is therefore

\[ 90e^{-0.05 \times 1} = 85.61 \]

(b) The probability that losses will be greater than $200 million is the probability that a normally distributed variable is greater than one standard deviation above the mean. This is 0.1587. The expected payoff in millions of dollars is therefore 0.1587 \times 100 = 15.87 and the value of the contract is

\[ 15.87e^{-0.05 \times 1} = 15.10 \]
CHAPTER 26
More on Models and Numerical Procedures

Notes for the Instructor

This chapter covers a number of nonstandard procedures that can be used to value derivatives. Each section of the chapter is independent of each other section. This means that instructors can teach some sections and omit others if they want to. The first three sections expose the student to some of the alternatives to Black-Scholes that provide a better fit than Black-Scholes to the volatility smiles that are encountered in practice (see Chapter 18). Section 26.4 deals with convertibles. (This material was moved to this chapter from the chapter on credit derivatives.) Sections 26.5 to 26.7 deal with a variety of numerical procedures for handling path-dependent options, barrier options, and options involving two correlated asset prices. The final section uses examples to explain two ways of valuing American options with Monte Carlo simulation.

Problems 26.19 to 26.23 all work well as assignments.

QUESTIONS AND PROBLEMS

Problem 26.1.
Confirm that the CEV model formulas satisfy put–call parity.

It follows immediately from the equations in Section 26.1 that

\[ p - c = Ke^{-rT} - S_0e^{-qT} \]

in all cases.

Problem 26.2.

Explain how you would use Monte Carlo simulation to sample paths for the asset price when Merton's jump diffusion model is used.

The probability of \( N \) jumps in time \( \Delta t \) is

\[ e^{-\lambda \Delta t} (\lambda \Delta t)^N \]

\[ \frac{N!} {N!} \]

When \( \Delta t \) is small we can ignore terms of order \((\Delta t)^2\) and higher so that the probability of no jumps is \( 1 - \lambda \Delta t \) and the probability of one jump is \( \lambda \Delta t \). During each time step of length \( \Delta t \) we first sample a random number between 0 and 1 to determine whether a jump takes place. Suppose for example that \( \lambda = 0.8 \) and \( \Delta t = 0.1 \) so that the probability of no
jumps is 0.92 and the probability of one jump is 0.08. If the random number is between 0 and 0.92 there is no jump; if it is between 0.92 and 1, there is one jump. If there is a jump we sample from the appropriate distribution to determine the size of the jump. The change in the asset price in time $\Delta t$ is then given by

$$\frac{\Delta S}{S} = (\mu - \lambda k)\Delta t + \sigma \varepsilon \sqrt{\Delta t} + Q$$

where $Q = 0$ if there is no jump and $Q$ is the size of the jump if a jump takes place. We can adjust this procedure to sample $\ln S$ rather than $S$ and to allow for more than one jump in time $\Delta t$.

**Problem 26.3.**

Confirm that Merton’s jump diffusion model satisfies put–call parity when the jump size is lognormal.

With the notation in the text the value of a call option, $c$ is

$$\sum_{n=0}^{\infty} \frac{e^{-\lambda' T} (\lambda' T)^n}{n!} c_n$$

where $c_n$ is the Black-Scholes price of a call option where the variance rate is

$$\sigma^2 + \frac{ns^2}{T}$$

and the risk-free rate is

$$r - \lambda k + \frac{n\gamma}{T}$$

where $\gamma = \ln(1 + k)$. Similarly the value of a put option $p$ is

$$\sum_{n=0}^{\infty} \frac{e^{-\lambda' T} (\lambda' T)^n}{n!} p_n$$

where $p_n$ is the Black-Scholes price of a put option with this variance rate and risk-free rate. It follows that

$$p - c = \sum_{n=0}^{\infty} \frac{e^{-\lambda' T} (\lambda' T)^n}{n!} (p_n - c_n)$$

From put–call parity

$$p_n - c_n = K e^{(-r+\lambda k)T} e^{-n\gamma} - S_0 e^{-qT}$$

Because

$$e^{-n\gamma} = (1 + k)^{-n}$$

it follows that

$$p - c = \sum_{n=0}^{\infty} \frac{e^{-\lambda' T + \lambda k T} (\lambda' T/(1 + k))^n}{n!} Ke^{-rT} - \sum_{n=0}^{\infty} \frac{e^{-\lambda' T} (\lambda' T)^n}{n!} S_0 e^{-qT}$$

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Using $\lambda' = \lambda(1 + k)$ this becomes

$$
\frac{1}{e^{\lambda'T}} \sum_{n=0}^{\infty} \frac{(\lambda'T)^n}{n!} K e^{-rT} - \frac{1}{e^{\lambda'T}} \sum_{n=0}^{\infty} \frac{(\lambda'T)^n}{n!} S_0 e^{-qT}
$$

From the expansion of the exponential function we get

$$
e^{\lambda'T} = \sum_{n=0}^{\infty} \frac{(\lambda'T)^n}{n!}
$$

$$
e^{\lambda'T} = \sum_{n=0}^{\infty} \frac{(\lambda'T)^n}{n!}
$$

Hence

$$p - c = Ke^{-rT} - S_0 e^{-qT}
$$

showing that put–call parity holds.

**Problem 26.4.**

Suppose that the volatility of an asset will be 20% from month 0 to month 6, 22% from month 6 to month 12, and 24% from month 12 to month 24. What volatility should be used in Black–Scholes to value a two-year option?

The average variance rate is

$$
\frac{6 \times 0.2^2 + 6 \times 0.22^2 + 12 \times 0.24^2}{24} = 0.0509
$$

The volatility used should be $\sqrt{0.0509} = 0.2256$ or 22.56%.

**Problem 26.5.**

Consider the case of Merton's jump diffusion model where jumps always reduce the asset price to zero. Assume that the average number of jumps per year is $\lambda$. Show that the price of a European call option is the same as in a world with no jumps except that the risk-free rate is $r + \lambda$ rather than $r$. Does the possibility of jumps increase or reduce the value of the call option in this case? (Hint: Value the option assuming no jumps and assuming one or more jumps. The probability of no jumps in time $T$ is $e^{-\lambda T}$.)

In a risk-neutral world the process for the asset price exclusive of jumps is

$$
\frac{dS}{S} = (r - q - \lambda k) dt + \sigma dz
$$

In this case $k = -1$ so that the process is

$$
\frac{dS}{S} = (r - q + \lambda) dt + \sigma dz
$$

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The asset behaves like a stock paying a dividend yield of $q - \lambda$. This shows that, conditional on no jumps, call price

$$S_0 e^{-(q-\lambda)T} N(d_1) - Ke^{-rT}$$

where

$$d_1 = \frac{\ln(S_0/K) + (r - q + \lambda + \sigma^2/2)T}{\sigma \sqrt{T}}$$

$$d_2 = d_1 - \sigma \sqrt{T}$$

There is a probability of $e^{-\lambda T}$ that there will be no jumps and a probability of $1 - e^{-\lambda T}$ that there will be one or more jumps so that the final asset price is zero. It follows that there is a probability of $e^{-\lambda T}$ that the value of the call is given by the above equation and $1 - e^{-\lambda T}$ that it will be zero. Because jumps have no systematic risk it follows that the value of the call option is

$$e^{-\lambda T}[S_0 e^{-(q-\lambda)T} N(d_1) - Ke^{-rT}]$$

or

$$S_0 e^{-qT} N(d_1) - Ke^{-(r+\lambda)T}$$

This is the required result. The value of a call option is an increasing function of the risk-free interest rate (see Chapter 9). It follows that the possibility of jumps increases the value of the call option in this case.

**Problem 26.6.**

At time zero the price of a non-dividend-paying stock is $S_0$. Suppose that the time interval between 0 and $T$ is divided into two subintervals of length $t_1$ and $t_2$. During the first subinterval, the risk-free interest rate and volatility are $r_1$ and $\sigma_1$, respectively. During the second subinterval, they are $r_2$ and $\sigma_2$, respectively. Assume that the world is risk neutral.

(a) Use the results in Chapter 13 to determine the stock price distribution at time $T$ in terms of $r_1$, $r_2$, $\sigma_1$, $\sigma_2$, $t_1$, $t_2$, and $S_0$.

(b) Suppose that $\bar{r}$ is the average interest rate between time zero and $T$ and that $\bar{\sigma}$ is the average variance rate between times zero and $T$. What is the stock price distribution as a function of $T$ in terms of $\bar{r}$, $\bar{\sigma}$, $T$, and $S_0$?

(c) What are the results corresponding to (a) and (b) when there are three subintervals with different interest rates and volatilities?

(d) Show that if the risk-free rate, $r$, and the volatility, $\sigma$, are known functions of time, the stock price distribution at time $T$ in a risk-neutral world is

$$\ln S_T \sim \phi \left[ \ln S_0 + \left( \bar{r} - \frac{\bar{\sigma}}{2} \right) T, VT \right]$$

where $\bar{r}$ is the average value of $r$, $\bar{\sigma}$ is equal to the average value of $\sigma^2$, and $S_0$ is the stock price today.
(a) Suppose that $S_1$ is the stock price at time $t_1$ and $S_T$ is the stock price at time $T$. From equation (13.3), it follows that in a risk-neutral world:

$$\ln S_1 - \ln S_0 \sim \phi \left[ \left( r_1 - \frac{\sigma_1^2}{2} \right) t_1, \sigma_1^2 t_1 \right]$$

$$\ln S_T - \ln S_1 \sim \phi \left[ \left( r_2 - \frac{\sigma_2^2}{2} \right) t_2, \sigma_2^2 t_2 \right]$$

Since the sum of two independent normal distributions is normal with mean equal to the sum of the means and variance equal to the sum of the variances

$$\ln S_T - \ln S_0 \sim \phi \left( r_1 t_1 + r_2 t_2 - \frac{\sigma_1^2 t_1}{2} - \frac{\sigma_2^2 t_2}{2}, \sigma_1^2 t_1 + \sigma_2^2 t_2 \right)$$

(b) Because

$$r_1 t_1 + r_2 t_2 = \bar{r} T$$

and

$$\sigma_1^2 t_1 + \sigma_2^2 t_2 = \bar{V} T$$

it follows that:

$$\ln S_T - \ln S_0 \sim \phi \left( \left( \bar{r} - \frac{\bar{V}}{2} \right) T, \bar{V} T \right)$$

(c) If $\sigma_i$ and $r_i$ are the volatility and risk-free interest rate during the $i$th subinterval ($i = 1, 2, 3$), an argument similar to that in (a) shows that:

$$\ln S_T - \ln S_0 \sim \phi \left( r_1 t_1 + r_2 t_2 + r_3 t_3 - \frac{\sigma_1^2 t_1}{2} - \frac{\sigma_2^2 t_2}{2} - \frac{\sigma_3^2 t_3}{2}, \sigma_1^2 t_1 + \sigma_2^2 t_2 + \sigma_3^2 t_3 \right)$$

where $t_1$, $t_2$ and $t_3$ are the lengths of the three subintervals. It follows that the result in (b) is still true.

(d) The result in (b) remains true as the time between time zero and time $T$ is divided into more subintervals, each having its own risk-free interest rate and volatility. In the limit, it follows that, if $r$ and $\sigma$ are known functions of time, the stock price distribution at time $T$ is the same as that for a stock with a constant interest rate and variance rate with the constant interest rate equal to the average interest rate and the constant variance rate equal to the average variance rate.

**Problem 26.7.**

Write down the equations for simulating the path followed by the asset price in the stochastic volatility model in equation (26.2) and (26.3).

The equations are:

$$S(t + \Delta t) = S(t) \exp[(r - q - V(t)/2)\Delta t + \epsilon_1 \sqrt{V(t)}\Delta t]$$

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\[ V(t + \Delta t) - V(t) = a[V_L - V(t)]\Delta t + \xi\epsilon V(t)\alpha\sqrt{\Delta t} \]

**Problem 26.8.**

"The IVF model does not necessarily get the evolution of the volatility surface correct." Explain this statement.

The IVF model is designed to match the volatility surface today. There is no guarantee that the volatility surface given by the model at future times will be the same as today—or that it will be even reasonable.

**Problem 26.9.**

"When interest rates are constant the IVF model correctly values any derivative whose payoff depends on the value of the underlying asset at only one time." Explain this statement.

The IVF model ensures that the risk-neutral probability distribution of the asset price at any future time conditional on its value today is correct (or at least consistent with the market prices of options). When a derivative's payoff depends on the value of the asset at only one time the IVF model therefore calculates the expected payoff from the asset correctly. The value of the derivative is the present value of the expected payoff. When interest rates are constant the IVF model calculates this present value correctly.

**Problem 26.10.**

Use a three-time-step tree to value an American lookback call option on a currency when the initial exchange rate is $1.6$, the domestic risk-free rate is $5\%$ per annum, the foreign risk-free interest rate is $8\%$ per annum, the exchange rate volatility is $15\%$, and the time to maturity is 18 months. Use the approach in Section 26.5.

In this case \( S_0 = 1.6, r = 0.05, r_f = 0.08, \sigma = 0.15, T = 1.5, \Delta t = 0.5 \). This means that

\[
\begin{align*}
    u &= e^{0.15\sqrt{0.5}} = 1.1119 \\
    d &= \frac{1}{u} = 0.8994 \\
    a &= e^{(0.05-0.08)\times0.5} = 0.9851 \\
    p &= \frac{a - d}{u - d} = 0.4033 \\
    1 - p &= 0.5967
\end{align*}
\]

The option pays off

\[ S_T - S_{\text{min}} \]

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The tree is shown in Figure S26.1. At each node, the upper number is the exchange rate, the middle number(s) are the minimum exchange rate(s) so far, and the lower number(s) are the value(s) of the option. The tree shows that the value of the option today is 0.131.

**Problem 26.11.**

*What happens to the variance-gamma model as the parameter \( v \) tends to zero?*

As \( v \) tends to zero the value of \( q \) becomes \( T \) with certainty. This can be demonstrated using the GAMMADIST function in Excel. By using a series expansion for the ln function we see that \( \omega \) becomes \(-\theta T\). In the limit the distribution of \( \ln S_T \) therefore has a mean of \( \ln S_0 + (r - q)T \) and a standard deviation of \( \sigma \sqrt{T} \) so that the model becomes geometric Brownian motion.

**Problem 26.12.**

*Use a three-time-step tree to value an American put option on the geometric average of the price of a non-dividend-paying stock when the stock price is $40, the strike price is*
$40, the risk-free interest rate is 10% per annum, the volatility is 35% per annum, and the time to maturity is three months. The geometric average is measured from today until the option matures.

In this case \( S_0 = 40, K = 40, r = 0.1, \sigma = 0.35, T = 0.25, \Delta t = 0.08333 \). This means that
\[
\begin{align*}
    u &= e^{0.35 \sqrt{0.08333}} = 1.1063 \\
    d &= \frac{1}{u} = 0.9039 \\
    a &= e^{0.1 \times 0.08333} = 1.008368 \\
    p &= \frac{a - d}{u - d} = 0.5161 \\
    1 - p &= 0.4839
\end{align*}
\]

The option pays off
\[
40 - \overline{S}
\]
where \( \overline{S} \) denotes the geometric average. The tree is shown in Figure 26.2. At each node, the upper number is the stock price, the middle number(s) are the geometric average(s), and the lower number(s) are the value(s) of the option. The geometric averages are calculated using the first, the last and all intermediate stock prices on the path. The tree shows that the value of the option today is $1.40.

**Problem 26.13.**

*Can the approach for valuing path dependent options in Section 26.5 be used for a two-year American-style option that provides a payoff equal to \( \max(S_{ave} - K, 0) \) where \( S_{ave} \) is the average asset price over the three months preceding exercise. Explain your answer.**

As mentioned in Section 26.4, for the procedure to work it must be possible to calculate the value of the path function at time \( \tau + \Delta t \) from the value of the path function at time \( \tau \) and the value of the underlying asset at time \( \tau + \Delta t \). When \( S_{ave} \) is calculated from time zero until the end of the life of the option (as in the example considered in Section 26.4) this condition is satisfied. When it is calculated over the last three months it is not satisfied. This is because, in order to update the average with a new observation on \( S \), it is necessary to know the observation on \( S \) from three months ago that is now no longer part of the average calculation.

**Problem 26.14.**

*Verify that the 6.492 number in Figure 26.4 is correct.*

We consider the situation where the average at node X is 53.83. If there is an up movement to node Y the new average becomes:
\[
\frac{53.83 \times 5 + 54.68}{6} = 53.97
\]
Interpolating, the value of the option at node Y when the average is 53.97 is

\[
\frac{(53.97 - 51.12) \times 8.635 + (54.26 - 53.97) \times 8.101}{54.26 - 51.12} = 8.586
\]

Similarly if there is a down movement the new average will be

\[
\frac{53.83 \times 5 + 45.72}{6} = 52.48
\]

In this case the option price is 4.416. The option price at node X when the average is 53.83 is therefore:

\[
8.586 \times 0.5056 + 4.416 \times 0.4944)e^{-0.1\times0.05} = 6.492
\]

**Problem 26.15.**

Examine the early exercise policy for the eight paths considered in the example in Section 26.8. What is the difference between the early exercise policy given by the least squares approach and the exercise boundary parameterization approach? Which gives a higher option price for the paths sampled?
Under the least squares approach we exercise at time $t = 1$ in paths 4, 6, 7, and 8. We exercise at time $t = 2$ for none of the paths. We exercise at time $t = 3$ for path 3. Under the exercise boundary parameterization approach we exercise at time $t = 1$ for paths 6 and 8. We exercise at time $t = 2$ for path 7. We exercise at time $t = 3$ for paths 3 and 4. For the paths sampled the exercise boundary parameterization approach gives a higher value for the option. However, it may be biased upward. As mentioned in the text, once the early exercise boundary has been determined in the exercise boundary parameterization approach a new Monte Carlo simulation should be carried out.

**Problem 26.16.**
Consider a European put option on a non-dividend paying stock when the stock price is $100, the strike price is $110, the risk-free rate is 5% per annum, and the time to maturity is one year. Suppose that the average variance rate during the life of an option has a 0.20 probability of being 0.06, a 0.5 probability of being 0.09, and a 0.3 probability of being 0.12. The volatility is uncorrelated with the stock price. Estimate the value of the option. Use DerivaGem.

If the average variance rate is 0.06, the value of the option is given by Black-Scholes with a volatility of $\sqrt{0.06} = 24.495\%$; it is 12.460. If the average variance rate is 0.09, the value of the option is given by Black-Scholes with a volatility of $\sqrt{0.09} = 30.000\%$; it is 14.655. If the average variance rate is 0.12, the value of the option is given by Black-Scholes with a volatility of $\sqrt{0.12} = 34.641\%$; it is 16.506. The value of the option is the Black-Scholes price integrated over the probability distribution of the average variance rate. It is

$$0.2 \times 12.460 + 0.5 \times 14.655 + 0.3 \times 16.506 = 14.77$$

**Problem 26.17.**
When there are two barriers how can a tree be designed so that nodes lie on both barriers?

Suppose that there are two horizontal barriers, $H_1$ and $H_2$, with $H_1 < H_2$ and that the underlying stock price follows geometric Brownian motion. In a trinomial tree, there are three possible movements in the asset’s price at each node: up by a proportional amount $u$; stay the same; and down by a proportional amount $d$ where $d = 1/u$. We can always choose $u$ so that nodes lie on both barriers. The condition that must be satisfied by $u$ is

$$H_2 = H_1 u^N$$

or

$$\ln H_2 = \ln H_1 + N \ln u$$

for some integer $N$.

When discussing trinomial trees in Section 19.4, the value suggested for $u$ was $e^{\sigma \sqrt{3 \Delta t}}$ so that $\ln u = \sigma \sqrt{3 \Delta t}$. In the situation considered here, a good rule is to choose $\ln u$ as
close as possible to this value, consistent with the condition given above. This means that we set

$$\ln u = \frac{\ln H_2 - \ln H_1}{N}$$

where

$$N = \text{int}\left[\frac{\ln H_2 - \ln H_1}{\sigma\sqrt{3\Delta t}} + 0.5\right]$$

and \(\text{int}(x)\) is the integral part of \(x\). This means that nodes are at values of the stock price equal to \(H_1, H_1 u, H_1 u^2, \ldots, H_1 u^N = H_2\).

Normally the trinomial stock price tree is constructed so that the central node is the initial stock price. In this case, it is unlikely that the current stock price happens to be \(H_1 u^i\) for some \(i\). To deal with this the first trinomial movement should be from the initial stock price to \(H_1 u^{i-1}, H_1 u^i\) and \(H_1 u^{i+1}\) where \(i\) is chosen so that \(H_1 u^i\) is closest to the current stock price. The probabilities on all branches of the tree are chosen, as usual, to match the first two moments of the stochastic process followed by the asset price. The approach works well except when the initial asset price is close to a barrier.

Problem 26.18.

Consider an 18-month zero-coupon bond with a face value of $100 that can be converted into five shares of the company's stock at any time during its life. Suppose that the current share price is $20, no dividends are paid on the stock, the risk-free rate for all maturities is 6% per annum with continuous compounding, and the share price volatility is 25% per annum. Assume that the default intensity is 3% per year and the recovery rate is 35%. The bond is callable at $110. Use a three-time-step tree to calculate the value of the bond. What is the value of the conversion option (net of the issuer's call option)?

In this case \(\Delta t = 0.5, \lambda = 0.03, \sigma = 0.25, r = 0.06\) and \(q = 0\) so that \(u = 1.1360, d = 0.8803, a = 1.0305, p_u = 0.6386, p_d = 0.3465\), and the probability on default branches is 0.0149. This leads to the tree shown in Figure S26.3. The bond is called at nodes B and D and this forces exercise. Without the call the value at node D would be 129.55, the value at node B would be 115.94, and the value at node A would be 105.18. The value of the call option to the bond issuer is therefore 105.18 - 103.72 = 1.46.
ASSIGNMENT QUESTIONS

Problem 26.19.
A new European-style lookback call option on a stock index has a maturity of nine months. The current level of the index is 400, the risk-free rate is 6% per annum, the dividend yield on the index is 4% per annum, and the volatility of the index is 20%. Use the approach in Section 26.5 to value the option and compare your answer to the result given by DerivaGem using the analytic valuation formula.

Using three-month time steps the tree parameters are $\Delta t = 0.254$, $u = 1.1052$, $d = 0.9048$, $a = 1.0050$, $p = 0.5000$. The tree is shown in Figure M26.1. The value of the lookback option is 40.47. (A more efficient procedure for giving the same result is in Technical Note 13. We construct a tree for $Y(t) = G(t)/S(t)$ where $G(t)$ is the minimum value of the index to date and $S(t)$ is the value of the index at time $t$. The tree is shown in Figure M26.2. It values the option in units of the stock index. This means that we value an instrument that pays off $1 - Y(t)$. The tree shows that the value of the option is
0.1019 units of the stock index or $400 \times 0.1019 = 40.47$ dollars, as given by Figure M26.1. DerivaGem shows that the value given by the analytic formula is 53.38. This is higher than the value given by the tree because the tree assumes that the stock price is observed only three times when the minimum is calculated.

![Figure M26.1](image)

**Figure M26.1** Tree for Problem 26.19.

**Problem 26.20.**

Suppose that the volatilities used to price a six-month currency option are as in Table 18.2. Assume that the domestic and foreign risk-free rates are 5% per annum and the current exchange rate is 1.00. Consider a bull spread that consists of a long position in a six-month call option with strike price 1.05 and a short position in a six-month call option with a strike price 1.10.

(a) What is the value of the spread?
(b) What single volatility if used for both options gives the correct value of the bull spread? (Use the DerivaGem Application Builder in conjunction with Goal Seek or Solver.)
(c) Does your answer support the assertion at the beginning of the chapter that the correct volatility to use when pricing exotic options can be counterintuitive?
(d) Does the IVF model give the correct price for the bull spread?

(a) The six-month call option with a strike price of 1.05 should be valued with a volatility of 13.4% and is worth 0.01829. The call option with a strike price of 1.10 should be
valued with a volatility of 14.3% and is worth 0.00959. The bull spread is therefore worth \(0.01829 - 0.00959 = 0.00870\).

(b) We now ask what volatility, if used to value both options, gives this price. Using the DerivaGem Application Builder in conjunction with Goal Seek we find that the answer is 11.42%.

(c) Yes, this does support the contention at the beginning of the chapter that the correct volatility for valuing exotic options can be counterintuitive. We might reasonably expect the volatility to be between 13.4% (the volatility used to value the first option) and 14.3% (the volatility used to value the second option). 11.42% is well outside this range. The reason why the volatility is relatively low is as follows. The option provides the same payoff as a regular option with a 1.05 strike price when the asset price is between 1.05 and 1.10 and a lower payoff when the asset price is over 1.10. The implied probability distribution of the asset price (see Figure 18.2) is less heavy than the lognormal distribution in the 1.05 to 1.10 range and more heavy than the lognormal distribution in the > 1.10 range. This means that using a volatility of 13.4% (which is the implied volatility of a regular option with a strike price of 1.05) will give a price than is too high.

(d) The bull spread provides a payoff at only one time. It is therefore correctly valued by the IVF model.

**Problem 26.21.**

Repeat the analysis in Section 26.8 for the put option example on the assumption that the strike price is 1.13. Use both the least squares approach and the exercise boundary parameterization approach.

Consider first the least squares approach. At the two-year point the option is in the
money for paths 1, 3, 4, 6, and 7. The five observations on $S$ are 1.08, 1.07, 0.97, 0.77, and 0.84. The five continuation values are $0, 0.10e^{-0.06}, 0.21e^{-0.06}, 0.23e^{-0.06}, 0.12e^{-0.06}$. The best fit continuation value is

$$-1.394 + 3.795S - 2.276S^2$$

The best fit continuation values for the five paths are 0.0495, 0.0605, 0.1454, 0.1785, and 0.1876. These show that the option should be exercised at the two-year point for all five paths. There are six paths at the one-year point for which the option is in the money. These are paths 1, 4, 5, 6, 7, and 8. The six observations on $S$ are 1.09, 0.93, 1.11, 0.76, 0.92, and 0.88. The six continuation values are $0.05e^{-0.06}, 0.16e^{-0.06}, 0.36e^{-0.06}, 0.29e^{-0.06}, \text{and } 0$. The best fit continuation value is

$$2.055 - 3.317S + 1.341S^2$$

The best fit continuation values for the six paths are 0.0327, 0.1301, 0.0253, 0.3088, 0.1385, and 0.1746. These show that the option should be exercised at the one-year point for paths 1, 4, 6, 7, and 8. The value of the option if not exercised at time zero is therefore

$$\frac{1}{8}(0.04e^{-0.06} + 0 + 0.06e^{-0.012} + 0.20e^{-0.06} + 0 + 0.37e^{-0.06} + 0.21e^{-0.06} + 0.25e^{-0.06})$$

or 0.133. Exercising at time zero would yield 0.13. The option should therefore not be exercised at time zero and its value is 0.133.

Consider next the exercise boundary parametrization approach. At time two years it is optimal to exercise when the stock price is 0.84 or below. At time one year it is optimal to exercise whenever the option is in the money. The value of the option assuming no early exercise at time zero is therefore

$$\frac{1}{8}(0.04e^{-0.06} + 0 + 0.10e^{-0.018} + 0.20e^{-0.06} + 0.02e^{-0.06} + 0.37e^{-0.06} + 0.21e^{-0.06} + 0.25e^{-0.06})$$

or 0.139. Exercising at time zero would yield 0.13. The option should therefore not be exercised at time zero. The value at time zero is 0.139. However, this tends to be high. As explained in the text, we should use one Monte Carlo simulation to determine the early exercise boundary. We should then carry out a new Monte Carlo simulation using the early exercise boundary to value the option.

**Problem 26.22.**

Consider the situation in Merton’s jump diffusion model where the underlying asset is a non-dividend paying stock. The average frequency of jumps is one per year. The average percentage jump size is 2% and the standard deviation of the logarithm of the percentage jump size is 20%. The stock price is 100, the risk-free rate is 5%, the volatility, $\sigma$ provided by the diffusion part of the process is 15%, and the time to maturity is six months. Use the DerivaGem Application Builder to calculate an implied volatility when the strike price
is 80, 90, 100, 110, and 120. What does the volatility smile or skew that you obtain imply about the probability distribution of the stock price.

The price of the option using Merton's model can be calculated using the first 20 terms in the formula in Section 26.1. For strike prices of 80, 90, 100, 110, and 120, the option prices are 22.64, 14.17, 7.67, 3.86, and 2.04 respectively. The implied volatilities are 26.00%, 23.64%, 22.85%, 23.65%, and 25.42%, respectively. The smile is similar to that for foreign currencies in Chapter 18. The probability distribution of the asset price in six months has heavier tails than the lognormal distribution.

Problem 26.23.

A three-year convertible bond with a face value of $100 has been issued by company ABC. It pays a coupon of $5 at the end of each year. It can be converted into ABCs equity at the end of the first year or at the end of the second year. At the end of the first year, it can be exchanged for 3.6 shares immediately after the coupon date. At the end of the second year it can be exchanged for 3.5 shares immediately after the coupon date. The current stock price is $25 and the stock price volatility is 25%. No dividends are paid on the stock. The risk-free interest rate is 5% with continuous compounding. The yield on bonds issued by ABC is 7% with continuous compounding and the recovery rate is 30%.

(a) Use a three-step tree to calculate the value of the bond

(b) How much is the conversion option worth?

(c) What difference does it make to the value of the bond and the value of the conversion option if the bond is callable any time within the first two years for $115?

(d) Explain how your analysis would change if there were a dividend payment of $1 on the equity at the six month, 18-month, and 30-month points. Detailed calculations are not required. Hint: Use equation (22.2) to estimate the default intensity.

In this case \( \Delta t = 1, \lambda = 0.02/0.7 = 0.02857, \sigma = 0.25, r = 0.05, q = 0, u = 1.2023, d = 0.8318, a = 1.0513, p_u = 0.6557, p_d = 0.3161, \) and the probability of a default is 0.0282. The calculations are shown in Figure M26.3. The values at the nodes include the value of the coupon paid just before the node is reached. The value of the convertible is 105.21. The value if there is no conversion is calculated by working out the present value of the coupons and principal at 7%. It is 94.12. The value of the conversion option is therefore 11.09. Calling at node D makes no difference because the bond will be converted at that node anyway. Calling at node B (just before the coupon payment) does make a difference. It reduces the value of the convertible at node B to $115. The value of the bond at node A is reduced by 2.34. This is a reduction in the value of the conversion option. A dividend payment would affect the way the tree is constructed as described in Chapter 19.
Figure M26.3  Tree for Problem 26.23
CHAPTER 27
Martingales and Measures

Notes for the Instructor

This chapter explains the equivalent martingale measure result. The material in this chapter is important for a full understanding of the way interest rate derivatives are priced in later chapters. A discussion of Black's model (Section 27.6) has been included. This makes the material in Chapter 28 flow more smoothly. For completeness Section 27.9 has been added.

Chapter 27 is more abstract and conceptual than other chapters. Not all instructors will choose to teach the material to undergraduate or even masters-level students.

The first part of the chapter explains the market price of risk. It explains the

\[ \mu - r = \lambda \sigma \]

equation for securities dependent on a single variable and

\[ \mu - r = \sum \lambda_i \sigma_i \]

for securities dependent on several variables.

The chapter then moves on to discuss martingales and measures. The general approach is to start by deriving results for a world where there is only one source of uncertainty and then point out that the results can be extended to the situation where there are many sources of uncertainty. I start by explaining that, when we use traditional risk-neutral valuation, we are setting the market price of risk to zero, but we do not have to do this. It is sometimes convenient to make other choices for the market price of risk.

A key result is that, if \( f \) and \( g \) are security prices and there is only one source of uncertainty, \( f/g \) is a martingale in a world where the market price of risk is the volatility of \( g \). This is proved in Section 27.3. I refer to a world where the market price of risk is the volatility of \( g \) as a world that is “forward risk neutral with respect to \( g \)”. When \( g \) is the money market account we get the traditional risk-neutral world. Other values of the numeraire \( g \) lead to other worlds.

A key result, concerned with the impact of a change in the numeraire security \( g \), is proved in Section 27.7. Many instructors who teach this chapter will choose to present this result without proof.

Problem 27.15 is a fairly straightforward application of ideas in the chapter. Problems 27.16 and 27.17 are appropriate for students with good math skills.
QUESTIONS AND PROBLEMS

Problem 27.1.

How is the market price of risk defined for a variable that is not the price of an investment asset?

The market price of risk for a variable that is not the price of a traded security is the market price of risk of a traded security whose price is instantaneously perfectly positively correlated with the variable.

Problem 27.2.

Suppose that the market price of risk for gold is zero. If the storage costs are 1% per annum and the risk-free rate of interest is 6% per annum, what is the expected growth rate in the price of gold? Assume that gold provides no income.

If its market price of risk is zero, gold must, after storage costs have been paid, provide an expected return equal to the risk-free rate of interest. In this case, the expected return after storage costs must be 6% per annum. It follows that the expected growth rate in the price of gold must be 7% per annum.

Problem 27.3.

Consider two securities both of which are dependent on the same market variable. The expected returns from the securities are 8% and 12%. The volatility of the first security is 15%. The instantaneous risk-free rate is 4%. What is the volatility of the second security?

The market price of risk is

$$\frac{\mu - r}{\sigma}$$

This is the same for both securities. From the first security we know it must be

$$\frac{0.08 - 0.04}{0.15} = 0.26667$$

The volatility, \( \sigma \) for the second security is given by

$$\frac{0.12 - 0.04}{\sigma} = 0.26667$$

The volatility is 30%.

Problem 27.4.

An oil company is set up solely for the purpose of exploring for oil in a certain small area of Texas. Its value depends primarily on two stochastic variables: the price of oil and the quantity of proven oil reserves. Discuss whether the market price of risk for the second of these two variables is likely to be positive, negative, or zero.

It can be argued that the market price of risk for the second variable is zero. This is because the risk is unsystematic, i.e., it is totally unrelated to other risks in the economy.
To put this another way, there is no reason why investors should demand a higher return for bearing the risk since the risk can be totally diversified away.

**Problem 27.5.**

*Deduce the differential equation for a derivative dependent on the prices of two non-dividend-paying traded securities by forming a riskless portfolio consisting of the derivative and the two traded securities.*

Suppose that the price, \( f \), of the derivative depends on the prices, \( S_1 \) and \( S_2 \), of two traded securities. Suppose further that:

\[
dS_1 = \mu_1 S_1 dt + \sigma_1 S_1 dz_1 \\
\]

\[
dS_2 = \mu_2 S_2 dt + \sigma_2 S_2 dz_2 \\
\]

where \( dz_1 \) and \( dz_2 \) are Wiener processes with correlation \( \rho \). From Itô's lemma [see equation (27A.3)]

\[
df = \left( \mu_1 S_1 \frac{\partial f}{\partial S_1} + \mu_2 S_2 \frac{\partial f}{\partial S_2} + \frac{\partial f}{\partial t} + \frac{1}{2} \sigma_1^2 S_1^2 \frac{\partial^2 f}{\partial S_1^2} + \frac{1}{2} \sigma_2^2 S_2^2 \frac{\partial^2 f}{\partial S_2^2} \right) dt + \sigma_1 S_1 \frac{\partial f}{\partial S_1} dz_1 + \sigma_2 S_2 \frac{\partial f}{\partial S_2} dz_2 \\
\]

To eliminate the \( dz_1 \) and \( dz_2 \) we choose a portfolio, \( \Pi \), consisting of

\[
-1: \text{ derivative} \\
+ \frac{\partial f}{\partial S_1}: \text{ first traded security} \\
+ \frac{\partial f}{\partial S_2}: \text{ second traded security} \\
\]

\[
\Pi = -f + \frac{\partial f}{\partial S_1} S_1 + \frac{\partial f}{\partial S_2} S_2 \\
\]

\[
d\Pi = -df + \frac{\partial f}{\partial S_1} dS_1 + \frac{\partial f}{\partial S_2} dS_2 \\
= -\left( \frac{\partial f}{\partial t} + \frac{1}{2} \sigma_1^2 S_1^2 \frac{\partial^2 f}{\partial S_1^2} + \frac{1}{2} \sigma_2^2 S_2^2 \frac{\partial^2 f}{\partial S_2^2} + \rho \sigma_1 \sigma_2 S_1 S_2 \frac{\partial^2 f}{\partial S_1 \partial S_2} \right) dt \\
\]

Since the portfolio is instantaneously risk-free it must instantaneously earn the risk-free rate of interest. Hence

\[
d\Pi = r \Pi dt \\
\]

Combining the above equations

\[
- \left[ \frac{\partial f}{\partial t} + \frac{1}{2} \sigma_1^2 S_1^2 \frac{\partial^2 f}{\partial S_1^2} + \frac{1}{2} \sigma_2^2 S_2^2 \frac{\partial^2 f}{\partial S_2^2} + \rho \sigma_1 \sigma_2 S_1 S_2 \frac{\partial^2 f}{\partial S_1 \partial S_2} \right] dt \\
= r \left[ -f + \frac{\partial f}{\partial S_1} S_1 + \frac{\partial f}{\partial S_2} S_2 \right] dt \\
\]

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so that:

\[
\frac{\partial f}{\partial t} + r S_1 \frac{\partial f}{\partial S_1} + r S_2 \frac{\partial f}{\partial S_2} + \frac{1}{2} \sigma_1^2 S_1 \frac{\partial^2 f}{\partial S_1^2} + \frac{1}{2} \sigma_2^2 S_2 \frac{\partial^2 f}{\partial S_2^2} + \rho \sigma_1 \sigma_2 S_1 S_2 \frac{\partial^2 f}{\partial S_1 \partial S_2} = rf
\]

**Problem 27.6.**
Suppose that an interest rate, \(x\), follows the process

\[
dx = a(x_0 - x) \, dt + c \sqrt{x} \, dz
\]

where \(a\), \(x_0\), and \(c\) are positive constants. Suppose further that the market price of risk for \(x\) is \(\lambda\). What is the process for \(x\) in the traditional risk-neutral world?

The process for \(x\) can be written

\[
\frac{dx}{x} = \frac{a(x_0 - x)}{x} \, dt + \frac{c}{\sqrt{x}} \, dz
\]

Hence the expected growth rate in \(x\) is:

\[
\frac{a(x_0 - x)}{x}
\]

and the volatility of \(x\) is

\[
\frac{c}{\sqrt{x}}
\]

In a risk neutral world the expected growth rate should be changed to

\[
\frac{a(x_0 - x)}{x} - \lambda \frac{c}{\sqrt{x}}
\]

so that the process is

\[
\frac{dx}{x} = \left[ \frac{a(x_0 - x)}{x} - \lambda \frac{c}{\sqrt{x}} \right] \, dt + \frac{c}{\sqrt{x}} \, dz
\]

i.e.

\[
dx = \left[ a(x_0 - x) - \lambda c \sqrt{x} \right] \, dt + c\sqrt{x} \, dz
\]

Hence the drift rate should be reduced by \(\lambda c \sqrt{x}\).

**Problem 27.7.**

Prove that when the security \(f\) provides income at rate \(q\) equation (27.9) becomes

\[
\mu + q - r = \lambda \sigma.
\]

(Hint: Form a new security, \(f^*\) that provides no income by assuming that all the income from \(f\) is reinvested in \(f\).)
As suggested in the hint we form a new security $f^*$ which is the same as $f$ except that all income produced by $f$ is reinvested in $f$. Assuming we start doing this at time zero, the relationship between $f$ and $f^*$ is

$$f^* = fe^{qt}$$

If $\mu^*$ and $\sigma^*$ are the expected return and volatility of $f^*$, Ito's lemma shows that

$$\mu^* = \mu + q$$
$$\sigma^* = \sigma$$

From equation (27.9)

$$\mu^* - r = \lambda \sigma^*$$

It follows that

$$\mu + q - r = \lambda \sigma$$

**Problem 27.8.**

Show that when $f$ and $g$ provide income at rates $q_f$ and $q_g$ respectively, equation (27.15) becomes

$$f_0 = g_0 e^{(q_f - q_g)T} E_g \left( \frac{f_T}{g_T} \right)$$

(Hint: Form new securities $f^*$ and $g^*$ that provide no income by assuming that all the income from $f$ is reinvested in $f$ and all the income in $g$ is reinvested in $g$.)

As suggested in the hint, we form two new securities $f^*$ and $g^*$ which are the same as $f$ and $g$ at time zero, but are such that income from $f$ is reinvested in $f$ and income from $g$ is reinvested in $g$. By construction $f^*$ and $g^*$ are non-income producing and their values at time $t$ are related to $f$ and $g$ by

$$f^* = f e^{q_f t} \quad \quad g^* = g e^{q_g t}$$

From Ito’s lemma, the securities $g$ and $g^*$ have the same volatility. We can apply the analysis given in Section 27.3 to $f^*$ and $g^*$ so that from equation (27.15)

$$f_0^* = g_0^* E_g \left( \frac{f_T}{g_T} \right)$$

or

$$f_0 = g_0 E_g \left( \frac{f_T e^{q_f T}}{g_T e^{q_g T}} \right)$$

or

$$f_0 = g_0 e^{(q_f - q_g)T} E_g \left( \frac{f_T}{g_T} \right)$$

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Problem 27.9.

"The expected future value of an interest rate in a risk-neutral world is greater than it is in the real world." What does this statement imply about the market price of risk for (a) an interest rate and (b) a bond price. Do you think the statement is likely to be true? Give reasons.

This statement implies that the interest rate has a negative market price of risk. Since bond prices and interest rates are negatively correlated, the statement implies that the market price of risk for a bond price is positive. The statement is reasonable. When interest rates increase, there is a tendency for the stock market to decrease. This implies that interest rates have negative systematic risk, or equivalently that bond prices have positive systematic risk.

Problem 27.10.

The variable $S$ is an investment asset providing income at rate $q$ measured in currency $A$. It follows the process

$$dS = \mu_S S dt + \sigma_S S dz$$

in the real world. Defining new variables as necessary, give the process followed by $S$, and the corresponding market price of risk, in

(a) A world that is the traditional risk-neutral world for currency $A$.
(b) A world that is the traditional risk-neutral world for currency $B$.
(c) A world that is forward risk neutral with respect to a zero-coupon currency $A$ bond maturing at time $T$.
(d) A world that is forward risk neutral with respect to a zero-coupon currency $B$ bond maturing at time $T$.

(a) In the traditional risk-neutral world the process followed by $S$ is

$$dS = (r - q) S dt + \sigma_S S dz$$

where $r$ is the instantaneous risk-free rate. The market price of $dz$-risk is zero.

(b) In the traditional risk-neutral world for currency $B$ the process is

$$dS = (r - q + \rho Q S \sigma_S \sigma_Q) S dt + \sigma_S S dz$$

where $Q$ is the exchange rate (units of $A$ per unit of $B$), $\sigma_Q$ is the volatility of $Q$ and $\rho QS$ is the coefficient of correlation between $Q$ and $S$. The market price of $dz$-risk is $\rho QS \sigma_Q$.

(c) In a world that is forward risk neutral with respect to a zero-coupon bond in currency $A$ maturing at time $T$

$$dS = (r - q + \sigma_p) S dt + \sigma_S S dz$$

where $\sigma_P$ is the bond price volatility. The market price of $dz$-risk is $\sigma_P$.
(d) In a world that is forward risk neutral with respect to a zero-coupon bond in currency B maturing at time $T$

$$dS = (r - q + \sigma_S \sigma_P + \rho_{FS} \sigma_S \sigma_F) S dt + \sigma_S S dz$$

where $F$ is the forward exchange rate, $\sigma_F$ is the volatility of $F$ (units of A per unit of B), and $\rho_{FS}$ is the correlation between $F$ and $S$. The market price of $dz$-risk is $\sigma_P + \rho_{FS} \sigma_F$.

**Problem 27.11.**

*Explain the difference between the way a forward interest rate is defined and the way the forward values of other variables such as stock prices, commodity prices, and exchange rates are defined.*

The forward value of a stock price, commodity price, or exchange rate is the delivery price in a forward contract that causes the value of the forward contract to be zero. A forward bond price is calculated in this way. However, a forward interest rate is the interest rate implied by the forward bond price.

**Problem 27.12.**

*Prove the result in Section 27.5 that when*

$$df = \left[ r + \sum_{i=1}^{n} \lambda_i \sigma_{f,i} \right] f dt + \sum_{i=1}^{n} \sigma_{f,i} f dz_i$$

*and*

$$dg = \left[ r + \sum_{i=1}^{n} \lambda_i \sigma_{g,i} \right] g dt + \sum_{i=1}^{n} \sigma_{g,i} g dz_i$$

*with the $dz_i$ uncorrelated, $f/g$ is a martingale for $\lambda_i = \sigma_{g,i}$.*

Equation (27A.4) in the appendix to Chapter 27 gives:

$$d \ln f = \left[ r + \sum_{i=1}^{n} (\lambda_i \sigma_{f,i} - \sigma_{f,i}^2 / 2) \right] dt + \sum_{i=1}^{n} \sigma_{f,i} dz_i$$

$$d \ln g = \left[ r + \sum_{i=1}^{n} (\lambda_i \sigma_{g,i} - \sigma_{g,i}^2 / 2) \right] dt + \sum_{i=1}^{n} \sigma_{g,i} dz_i$$

*so that*

$$d \ln \frac{f}{g} = d \ln f - d \ln g = \left[ \sum_{i=1}^{n} (\lambda_i \sigma_{f,i} - \lambda_i \sigma_{g,i} - \sigma_{f,i}^2 / 2 + \sigma_{g,i}^2 / 2) \right] dt + \sum_{i=1}^{n} (\sigma_{f,i} - \sigma_{g,i}) dz_i$$

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Applying Ito's lemma again

\[ \frac{df}{g} = \frac{f}{g} \left[ \sum_{i=1}^{n} (\lambda_i \sigma_{f,i} - \lambda_i \sigma_{g,i} - \sigma^2_{f,i}/2 + \sigma^2_{g,i}/2) + (\sigma_{f,i} - \sigma_{g,i})^2 \right] dt + \frac{f}{g} \sum_{i=1}^{n} (\sigma_{f,i} - \sigma_{g,i})dz_i \]

When \( \lambda_i = \sigma_{g,i} \) the coefficient of \( dt \) is zero and \( f/g \) is a martingale.

**Problem 27.13.**

Prove equation (27.33) in Section 27.7.

Consider the case where \( v \) depends on \( m \) traded securities \( f_1, f_2, \ldots, f_m \) and the \( i \)th component of the volatility of \( f_j \) is \( \sigma_{i,j} \). With the notation in Section 27.7 when the numeraire changes from \( g \) to \( h \) the expected growth rate of \( f_j \) changes by

\[ \sum_{i=1}^{n} (\sigma_{h,i} - \sigma_{g,i})\sigma_{i,j} \]

Equation (27A.5) shows that the drift rate \( v \) changes by

\[ \sum_{j=1}^{m} \frac{\partial v}{\partial f_j} \sum_{i=1}^{n} (\sigma_{h,i} - \sigma_{g,i})\sigma_{i,j}f_j \]

The \( i \)th component of the volatility of \( v \), \( \sigma_{v,i} \), is from equation (27A.5) given by

\[ v\sigma_{v,i} = \sum_{j=1}^{m} \frac{\partial v}{\partial f_j} \sigma_{i,j}f_j \]

so that the drift rate of \( v \) changes by

\[ \sum_{i=1}^{n} (\sigma_{h,i} - \sigma_{g,i})v\sigma_{v,i} \]

This is the same as saying that the growth rate of \( v \) changes by

\[ \sum_{i=1}^{n} (\sigma_{h,i} - \sigma_{g,i})\sigma_{v,i} \]

and proves equation (27.33).

**Problem 27.14.**

Show that when \( w = h/g \) and \( h \) and \( g \) are each dependent on \( n \) Wiener processes, the \( i \)th component of the volatility of \( w \) is the \( i \)th component of the volatility of \( h \) minus the \( i \)th component of the volatility of \( g \). Use this to prove the result that if \( \sigma_{V} \) is the volatility
of $U$ and $\sigma_V$ is the volatility of $V$ then the volatility of $U/V$ is $\sqrt{\sigma_U^2 + \sigma_V^2 - 2 \rho \sigma_U \sigma_V}$.  
(HINT Use the result in footnote 7.)

Equation (27A.4) in the appendix to Chapter 27 gives:

$$d \ln h = \ldots + \sum_{i=1}^{n} \sigma_{h,i} dz_i$$

$$d \ln g = \ldots + \sum_{i=1}^{n} \sigma_{g,i} dz_i$$

so that

$$d \ln \frac{h}{g} = \ldots + \sum_{i=1}^{n} (\sigma_{h,i} - \sigma_{g,i}) dz_i$$

Applying Ito’s lemma again

$$d \ln \frac{h}{g} = \ldots + \frac{h}{g} \sum_{i=1}^{n} (\sigma_{h,i} - \sigma_{g,i}) dz_i$$

This proves the result.

ASSIGNMENT QUESTIONS

Problem 27.15.

A security’s price is positively dependent on two variables: the price of copper and the yen-dollar exchange rate. Suppose that the market price of risk for these variables is 0.5 and 0.1, respectively. If the price of copper were held fixed, the volatility of the security would be 8% per annum; if the yen-dollar exchange rate were held fixed, the volatility of the security would be 12% per annum. The risk-free interest rate is 7% per annum. What is the expected rate of return from the security? If the two variables are uncorrelated with each other, what is the volatility of the security?

Suppose that $S$ is the security price and $\mu$ is the expected return from the security. Then:

$$\frac{dS}{S} = \mu dt + \sigma_1 dz_1 + \sigma_2 dz_2$$

where $dz_1$ and $dz_2$ are Wiener processes, $\sigma_1 dz_1$ is the component of the risk in the return attributable to the price of copper and $\sigma_2 dz_2$ is the component of the risk in the return attributable to the yen-dollar exchange rate.

If the price of copper is held fixed, $dz_1 = 0$ and:

$$\frac{dS}{S} = \mu dt + \sigma_2 dz_2$$
Hence $\sigma_2$ is 8% per annum or 0.08. If the yen-dollar exchange rate is held fixed, $dz_2 = 0$ and:

$$\frac{dS}{S} = \mu dt + \sigma_1 dz_1$$

Hence $\sigma_1$ is 12% per annum or 0.12.

From equation (27.13)

$$\mu - r = \lambda_1 \sigma_1 + \lambda_2 \sigma_2$$

where $\lambda_1$ and $\lambda_2$ are the market prices of risk for copper and the yen-$\$ exchange rate. In this case, $r = 0.07$, $\lambda_1 = 0.5$ and $\lambda_2 = 0.1$. Therefore

$$\mu - 0.07 = 0.5 \times 0.12 + 0.1 \times 0.08$$

so that

$$\mu = 0.138$$

i.e., the expected return is 13.8% per annum.

If the two variables affecting $S$ are uncorrelated, we can use the result that the sum of normally distributed variables is normal with variance of the sum equal to the sum of the variances. This leads to:

$$\sigma_1 dz_1 + \sigma_2 dz_2 = \sqrt{\sigma_1^2 + \sigma_2^2} dz_3$$

where $dz_3$ is a Wiener process. Hence the process for $S$ becomes:

$$\frac{dS}{S} = \mu dt + \sqrt{\sigma_1^2 + \sigma_2^2} dz_3$$

If follows that the volatility of $S$ is $\sqrt{\sigma_1^2 + \sigma_2^2}$ or 14.4% per annum.

**Problem 27.16.**

Suppose that the price of a zero-coupon bond maturing at time $T$ follows the process

$$dP(t, T) = \mu_P P(t, T) dt + \sigma_P P(t, T) dz$$

and the price of a derivative dependent on the bond follows the process

$$df = \mu_f f dt + \sigma_f f dz$$

Assume only one source of uncertainty and that $f$ provides no income.

(a) What is the forward price, $F$, of $f$ for a contract maturing at time $T$?

(b) What is the process followed by $F$ in a world that is forward risk neutral with respect to $P(t, T)$?

(c) What is the process followed by $F$ in the traditional risk-neutral world?

(d) What is the process followed by $f$ in a world that is forward risk neutral with respect to a bond maturing at time $T^*$ where $T^* \neq T$? Assume that $\sigma^*_P$ is the volatility of this bond
(a) The no-arbitrage arguments in Chapter 5 show that

\[ F(t) = \frac{f(t)}{P(t,T)} \]

(b) From Ito's lemma:

\[ d\ln P = (\mu_P - \sigma_P^2/2) dt + \sigma_P dz \]
\[ d\ln f = (\mu_f - \sigma_f^2/2) dt + \sigma_f dz \]

Therefore

\[ d\ln \frac{f}{P} = d(\ln f - \ln P) = (\mu_f - \sigma_f^2/2 - \mu_P + \sigma_P^2/2) dt + (\sigma_f - \sigma_P) dz \]

so that

\[ d\frac{f}{P} = (\mu_f - \mu_P + \sigma_P^2 - \sigma_f \sigma_P) \frac{f}{P} dt + (\sigma_f - \sigma_P) \frac{f}{P} dz \]

or

\[ dF = (\mu_f - \mu_P + \sigma_P^2 - \sigma_f \sigma_P) F dt + (\sigma_f - \sigma_P) F dz \]

In a world that is forward risk neutral with respect to \(P(t,T)\), \(F\) has zero drift. The process for \(F\) is

\[ dF = (\sigma_f - \sigma_P) F dz \]

(c) In the traditional risk-neutral world, \(\mu_f = \mu_P = r\) where \(r\) is the short-term risk-free rate and

\[ dF = (\sigma_P^2 - \sigma_f \sigma_P) F dt + (\sigma_f - \sigma_P) F dz \]

Note that the answers to parts (b) and (c) are consistent with the market price of risk being zero in (c) and \(\sigma_P\) in (b). When the market price of risk is \(\sigma_P\), \(\mu_f = r + \sigma_f \sigma_P\) and \(\mu_P = r + \sigma_P^2\).

(d) In a world that is forward risk-neutral with respect to a bond maturing at time \(T^*\), \(\mu_P = r + \sigma_P^2 \sigma_P\) and \(\mu_f = r + \sigma_P^2 \sigma_f\) so that

\[ dF = [\sigma_P^2 - \sigma_f \sigma_P + \sigma_P^2 (\sigma_f - \sigma_P)] F dt + (\sigma_f - \sigma_P) F dz \]

or

\[ dF = (\sigma_f - \sigma_P) (\sigma_P^2 - \sigma_P) F dt + (\sigma_f - \sigma_P) F dz \]

Problem 27.17.

Consider a variable that is not an interest rate

(a) In what world is the futures price of the variable a martingale

(b) In what world is the forward price of the variable a martingale

(c) Defining variables as necessary derive an expression for the difference between the drift of the futures price and the drift of the forward price in the traditional risk-neutral world
(d) Show that your result is consistent with the points made in Section 5.8 about the circumstances when the futures price is above the forward price.

(a) The futures price is a martingale in the traditional risk-neutral world

(b) The forward price for a contract maturing at time $T$ is a martingale is a world that is forward risk neutral with respect to $P(t, T)$

(c) Define $\sigma_P$ as the volatility of $P(t, T)$ and $\sigma_F$ as the volatility of the forward price. The forward rate has zero drift in a world that is forward risk neutral with respect to $P(t, T)$. When we move from the traditional world to a world that is forward risk neutral with respect to $P(t, T)$ the volatility of the numeraire ratio is $\sigma_P$ and the drift increases by $\rho_{PF}\sigma_P\sigma_F$ where $\rho_{PF}$ is the correlation between $P(t, T)$ and the forward price. It follows that the drift of the forward price in the traditional risk neutral world is $-\rho_{PF}\sigma_P\sigma_F$. The drift of the futures price is zero in the traditional risk neutral world. It follows that the excess of the drift of the futures price over the forward price is $\rho_{PF}\sigma_P\sigma_F$.

(d) $P$ is inversely correlated with interest rates. It follows that when the correlation between interest rates and $F$ is positive the futures price has a lower drift than the forward price. The futures and forward prices are the same at maturity. It follows that the futures price is above the forward price prior to maturity. This is consistent with Section 5.8. Similarly when the correlation between interest rates and $F$ is negative the future price is below the forward price prior to maturity.
CHAPTER 28
Interest Rate Derivatives:
The Standard Market Models

Notes for the Instructor

This chapter explains how the market uses Black's model (which was originally de-
veloped for options on commodity futures) can be used for three of the most popular
over-the-counter interest rate options: European bond options, caps/floors, and European
swap options. The implied volatilities quoted for these products (see, for example, Tables
28.1 and 28.2) are the volatilities implied by Black's model. Day count conventions are
important in interest rate derivatives. In general, I find it best to initially ignore day
count conventions when products are being explained (or, to be more precise, to assume
actual/actual and that days are divisible). Later when students are comfortable with a
product, the impact of day count conventions can be explained. The chapter follows this
approach. There are sections on day count conventions after products have been explained.

The chapter uses the material in Chapter 27 to show that each application of Black's
model is internally consistent. These parts of the chapter can be skipped if Chapter 27
has not been covered.

I spend some time discussing each of the three products (European bond options,
caps/floors, and European swap options). In addition to the pricing formulas I explain
a number of market conventions. For example, (a) the yield volatility quoted for bond
options is usually converted to a price volatility, using the duration relationship, before
Black's model is used; (b) there is no payoff from a cap on the first reset date; (c) a 3×5
swap option is a three year option to enter into a swap which will last a further five years.

Problems 28.21 to 28.25 are fairly straightforward assignment questions. 28.23, 28.24,
and 28.25 require the use of DerivaGem.

QUESTIONS AND PROBLEMS

Problem 28.1.

A company caps three-month LIBOR at 10% per annum. The principal amount is $20
million. On a reset date, three-month LIBOR is 12% per annum. What payment would
this lead to under the cap? When would the payment be made?

An amount

\[ \$20,000,000 \times 0.02 \times 0.25 = \$100,000 \]

would be paid out 3 months later.
Problem 28.2.

Explain why a swap option can be regarded as a type of bond option.

A swap option (or swaption) is an option to enter into an interest rate swap at a certain time in the future with a certain fixed rate being used. An interest rate swap can be regarded as the exchange of a fixed-rate bond for a floating-rate bond. A swaption is therefore the option to exchange a fixed-rate bond for a floating-rate bond. The floating-rate bond will be worth its face value at the beginning of the life of the swap. The swaption is therefore an option on a fixed-rate bond with the strike price equal to the face value of the bond.

Problem 28.3.

Use the Black's model to value a one-year European put option on a 10-year bond. Assume that the current value of the bond is $125, the strike price is $110, the one-year interest rate is 10% per annum, the bond’s forward price volatility is 8% per annum, and the present value of the coupons to be paid during the life of the option is $10.

In this case, $F_0 = (125 - 10)e^{0.1 \times 1} = 127.09$, $K = 110$, $P(0, T) = e^{-0.1 \times 1}$, $\sigma_B = 0.08$, and $T = 1.0$.

$$d_1 = \frac{\ln(127.09/110) + (0.08^2/2)}{0.08} = 1.8456$$
$$d_2 = d_1 - 0.08 = 1.7656$$

From equation (28.2) the value of the put option is

$$110e^{-0.1 \times 1}N(-1.7656) - 127.09e^{-0.1 \times 1}N(-1.8456) = 0.12$$

or $0.12$.

Problem 28.4.

Explain carefully how you would use (a) spot volatilities and (b) flat volatilities to value a five-year cap.

When spot volatilities are used to value a cap, a different volatility is used to value each caplet. When flat volatilities are used, the same volatility is used to value each caplet within a given cap. Spot volatilities are a function of the maturity of the caplet. Flat volatilities are a function of the maturity of the cap.

Problem 28.5.

Calculate the price of an option that caps the three-month rate, starting in 15 months time, at 13% (quoted with quarterly compounding) on a principal amount of $1,000. The forward interest rate for the period in question is 12% per annum (quoted with quarterly compounding), the 18-month risk-free interest rate (continuously compounded) is 11.5% per annum, and the volatility of the forward rate is 12% per annum.

In this case $L = 1000$, $\delta_k = 0.25$, $F_k = 0.12$, $R_k = 0.13$, $r = 0.115$, $\sigma_k = 0.12$, $t_k = 1.25$, $P(0, t_{k+1}) = 0.8416$.

$$L\delta_k = 250$$

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\[ d_1 = \frac{\ln(0.12/0.13) + 0.12^2 \times 1.25/2}{0.12\sqrt{1.25}} = -0.5295 \]
\[ d_2 = -0.5295 - 0.12\sqrt{1.25} = -0.6637 \]

The value of the option is
\[ 250 \times 0.8416 \times [0.12N(-0.5295) - 0.13N(-0.6637)] \]
\[ = 0.59 \]
or $0.59.

**Problem 28.6.**

A bank uses Black's model to price European bond options. Suppose that an implied price volatility for a 5-year option on a bond maturing in 10 years is used to price a 9-year option on the bond. Would you expect the resultant price to be too high or too low? Explain.

The implied volatility measures the standard deviation of the logarithm of the bond price at the maturity of the option divided by the square root of the time to maturity. In the case of a five year option on a ten year bond, the bond has five years left at option maturity. In the case of a nine year option on a ten year bond it has one year left. The standard deviation of a one year bond price observed in nine years can be normally be expected to be considerably less than that of a five year bond price observed in five years. (See Figure 28.1.) We would therefore expect the price to be too high.

**Problem 28.7.**

Calculate the value of a four-year European call option on bond that will mature five years from today using Black's model. The five-year cash bond price is $105, the cash price of a four-year bond with the same coupon is $102, the strike price is $100, the four-year risk-free interest rate is 10% per annum with continuous compounding, and the volatility for the bond price in four years is 2% per annum.

The present value of the principal in the four year bond is \(100e^{-4\times0.1} = 67.032\). The present value of the coupons is, therefore, \(102 - 67.032 = 34.968\). This means that the forward price of the five-year bond is

\[ (105 - 34.968)e^{4\times0.1} = 104.475 \]

The parameters in Black's model are therefore \(F_0 = 104.475, K = 100, r = 0.1, T = 4, \) and \(\sigma = 0.02\).

\[ d_1 = \frac{\ln 1.04475 + 0.5 \times 0.02^2 \times 4}{0.02\sqrt{4}} = 1.1144 \]
\[ d_2 = d_1 - 0.02\sqrt{4} = 1.0744 \]

The price of the European call is

\[ e^{-0.1 \times 4}[104.475N(1.1144) - 100N(1.0744)] = 3.19 \]
or $3.19.

**Problem 28.8.**

*If the yield volatility for a five-year put option on a bond maturing in 10 years time is specified as 22%, how should the option be valued? Assume that, based on today's interest rates the modified duration of the bond at the maturity of the option will be 4.2 years and the forward yield on the bond is 7%.*

The option should be valued using Black's model in equation (28.2) with the bond price volatility being

\[ 4.2 \times 0.07 \times 0.22 = 0.0647 \]

or 6.47%.

**Problem 28.9.**

*What other instrument is the same as a five-year zero-cost collar where the strike price of the cap equals the strike price of the floor? What does the common strike price equal?*

A 5-year zero-cost collar where the strike price of the cap equals the strike price of the floor is the same as an interest rate swap agreement to receive floating and pay a fixed rate equal to the strike price. The common strike price is the swap rate. Note that the swap is actually a forward swap that excludes the first exchange. (See Business Snapshot 28.1)

**Problem 28.10.**

*Derive a put-call parity relationship for European bond options.*

There are two way of expressing the put-call parity relationship for bond options. The first is in terms of bond prices:

\[ c + I + Ke^{-RT} = p + B \]

where \( c \) is the price of a European call option, \( p \) is the price of the corresponding European put option, \( I \) is the present value of the bond coupon payments during the life of the option, \( K \) is the strike price, \( T \) is the time to maturity, \( B \) is the bond price, and \( R \) is the risk-free interest rate for a maturity equal to the life of the options. To prove this we can consider two portfolios. The first consists of a European put option plus the bond; the second consists of the European call option, and an amount of cash equal to the present value of the coupons plus the present value of the strike price. Both can be seen to be worth the same at the maturity of the options.

The second way of expressing the put-call parity relationship is

\[ c + Ke^{-RT} = p + F_0 e^{-RT} \]

where \( F_0 \) is the forward bond price. This can also be proved by considering two portfolios. The first consists of a European put option plus a forward contract on the bond plus the present value of the forward price; the second consists of a European call option plus the
present value of the strike price. Both can be seen to be worth the same at the maturity of the options.

**Problem 28.11.**  
*Derive a put-call parity relationship for European swap options.*

The put-call parity relationship for European swap options is

\[ c + V = p \]

where \( c \) is the value of a call option to pay a fixed rate of \( s_K \) and receive floating, \( p \) is the value of a put option to receive a fixed rate of \( s_K \) and pay floating, and \( V \) is the value of the forward swap underlying the swap option where \( s_K \) is received and floating is paid. This can be proved by considering two portfolios. The first consists of the put option; the second consists of the call option and the swap. Suppose that the actual swap rate at the maturity of the options is greater than \( s_K \). The call will be exercised and the put will not be exercised. Both portfolios are then worth zero. Suppose next that the actual swap rate at the maturity of the options is less than \( s_K \). The put option is exercised and the call option is not exercised. Both portfolios are equivalent to a swap where \( s_K \) is received and floating is paid. In all states of the world the two portfolios are worth the same at time \( T \). They must therefore be worth the same today. This proves the result.

**Problem 28.12.**  
*Explain why there is an arbitrage opportunity if the implied Black (flat) volatility of a cap is different from that of a floor. Do the broker quotes in Table 28.1 present an arbitrage opportunity?*

Suppose that the cap and floor have the same strike price and the same time to maturity. The following put-call parity relationship must hold:

\[ \text{cap} + \text{swap} = \text{floor} \]

where the swap is an agreement to receive the cap rate and pay floating over the whole life of the cap/floor. If the implied Black volatilities for the cap equals that for the floor, the Black formulas show that this relationship holds. In other circumstances it does not hold and there is an arbitrage opportunity. The broker quotes in Table 28.1 do not present an arbitrage opportunity because the cap offer is always higher than the floor bid and the floor offer is always higher than the cap bid.

**Problem 28.13.**  
*When a bond’s price is lognormal can the bond’s yield be negative? Explain your answer.*

Yes. If a zero-coupon bond price at some future time is lognormal, there is some chance that the price will be above par. This in turn implies that the yield to maturity on the bond is negative.

What is the value of a European swap option that gives the holder the right to enter into a 3-year annual-pay swap in four years where a fixed rate of 5% is paid and LIBOR is received? The swap principal is $10 million. Assume that the yield curve is flat at 5% per annum with annual compounding and the volatility of the swap rate is 20%. Compare your answer to that given by DerivaGem.

In equation (28.10), $L = 10,000,000$, $s_K = 0.05$, $s_0 = 0.05$, $d_1 = 0.2\sqrt{4/2} = 0.2$, $d_2 = -.2$, and

$$A = \frac{1}{1.05^5} + \frac{1}{1.05^6} + \frac{1}{1.05^7} = 2.2404$$

The value of the swap option (in millions of dollars) is

$$10 \times 2.2404\left[0.05N(0.2) - 0.05N(-0.2)\right] = 0.178$$

This is the same as the answer given by DerivaGem. (For the purposes of using the DerivaGem software note that the interest rate is 4.879% with continuous compounding for all maturities.)

Problem 28.15.

Suppose that the yield, $R$, on a zero-coupon bond follows the process

$$dR = \mu dt + \sigma dz$$

where $\mu$ and $\sigma$ are functions of $R$ and $t$, and $dz$ is a Wiener process. Use Ito's lemma to show that the volatility of the zero-coupon bond price declines to zero as it approaches maturity.

The price of the bond at time $t$ is $e^{-R(T-t)}$ where $T$ is the time when the bond matures. Using Itô's lemma the volatility of the bond price is

$$\sigma \frac{\partial}{\partial R} e^{-R(T-t)} = -\sigma (T-t) e^{-R(T-t)}$$

This tends to zero as $t$ approaches $T$.

Problem 28.16.

Carry out a manual calculation to verify the option prices in Example 28.2.

The cash price of the bond is

$$4e^{-0.05\times0.50} + 4e^{-0.05\times1.00} + \ldots + 4e^{-0.05\times10} + 100e^{-0.05\times10} = 122.82$$

As there is no accrued interest this is also the quoted price of the bond. The interest paid during the life of the option has a present value of

$$4e^{-0.05\times0.5} + 4e^{-0.05\times1} + 4e^{-0.05\times1.5} + 4e^{-0.05\times2} = 15.04$$
The forward price of the bond is therefore

\[ (122.82 - 15.04)e^{0.05 \times 2.25} = 120.61 \]

The duration of the bond at option maturity is

\[
\frac{0.25 \times 4e^{-0.05 \times 0.25} + \ldots + 7.75 \times 4e^{-0.05 \times 7.75} + 7.75 \times 100e^{-0.05 \times 7.75}}{4e^{-0.05 \times 0.25} + 4e^{-0.05 \times 0.75} + \ldots + 4e^{-0.05 \times 7.75} + 100e^{-0.05 \times 7.75}}
\]

or 5.99. The bond price volatility is therefore 5.99 \times 0.05 \times 0.2 = 0.0599. We can therefore value the bond option using Black’s model with \( F_0 = 120.61 \), \( P(0, 2.25) = e^{-0.05 \times 2.25} = 0.8936 \), \( \sigma = 5.99\% \), and \( T = 2.25 \). When the strike price is the cash price \( K = 115 \) and the value of the option is 1.78. When the strike price is the quoted price \( K = 117 \) and the value of the option is 2.41.

**Problem 28.17.**

Suppose that the 1-year, 2-year, 3-year, 4-year and 5-year zero rates are 6%, 6.4%, 6.7%, 6.9%, and 7%. The price of a 5-year semiannual cap with a principal of $100 at a cap rate of 8% is $3. Use DerivaGem to determine

a. The 5-year flat volatility for caps and floors
b. The floor rate in a zero-cost 5-year collar when the cap rate is 8%

We choose the Caps and Swap Options worksheet of DerivaGem and choose Cap/Floor as the Underlying Type. We enter the 1-, 2-, 3-, 4-, 5-year zero rates as 6%, 6.4%, 6.7%, 6.9%, and 7.0% in the Term Structure table. We enter Semiannual for the Settlement Frequency, 100 for the Principal, 0 for the Start (Years), 5 for the End (Years), 8% for the Cap/Floor Rate, and $3 for the Price. We select Black-European as the Pricing Model and choose the Cap button. We check the Imply Volatility box and Calculate. The implied volatility is 24.79%. We then uncheck Implied Volatility, select Floor, check Imply Breakeven Rate. The floor rate that is calculated is 6.71%. This is the floor rate for which the floor is worth $3. A collar when the floor rate is 6.71% and the cap rate is 8% has zero cost.

**Problem 28.18.**

Show that \( V_1 + f = V_2 \) where \( V_1 \) is the value of a swap option to pay a fixed rate of \( s_K \) and receive LIBOR between times \( T_1 \) and \( T_2 \), \( f \) is the value of a forward swap to receive a fixed rate of \( s_K \) and pay LIBOR between times \( T_1 \) and \( T_2 \), and \( V_2 \) is the value of a swap option to receive a fixed rate of \( s_K \) between times \( T_1 \) and \( T_2 \). Deduce that \( V_1 = V_2 \) when \( s_K \) equals the current forward swap rate.

We prove this result by considering two portfolios. The first consists of the swap option to receive \( s_K \); the second consists of the swap option to pay \( s_K \) and the forward swap. Suppose that the actual swap rate at the maturity of the options is greater than \( s_K \). The swap option to pay \( s_K \) will be exercised and the swap option to receive \( s_K \) will not be exercised. Both portfolios are then worth zero since the swap option to pay \( s_K \) is neutralized by the forward swap. Suppose next that the actual swap rate at the maturity
of the options is less than $S_K$. The swap option to receive $s_K$ is exercised and the swap option to pay $s_K$ is not exercised. Both portfolios are then equivalent to a swap where $s_K$ is received and floating is paid. In all states of the world the two portfolios are worth the same at time $T_1$. They must therefore be worth the same today. This proves the result. When $s_K$ equals the current forward swap rate $f = 0$ and $V_1 = V_2$. A swap option to pay fixed is therefore worth the same as a similar swap option to receive fixed when the fixed rate in the swap option is the forward swap rate.

**Problem 28.19.**

Suppose that zero rates are as in Problem 28.17. Use DerivaGem to determine the value of an option to pay a fixed rate of 6% and receive LIBOR on a five-year swap starting in one year. Assume that the principal is $100 million, payments are exchanged semiannually, and the swap rate volatility is 21%.

We choose the Caps and Swap Options worksheet of DerivaGem and choose Swap Option as the Underlying Type. We enter 100 as the Principal, 1 as the Start (Years), 6 as the End (Years), 6% as the Swap Rate, and Semiannual as the Settlement Frequency. We choose Black-European as the pricing model, enter 21% as the Volatility and check the Pay Fixed button. We do not check the Imply Breakeven Rate and Imply Volatility boxes. The value of the swap option is 5.63.

**Problem 28.20.**

Describe how you would (a) calculate cap flat volatilities from cap spot volatilities and (b) calculate cap spot volatilities from cap flat volatilities.

(a) To calculate flat volatilities from spot volatilities we choose a strike rate and use the spot volatilities to calculate caplet prices. We then sum the caplet prices to obtain cap prices and imply flat volatilities from Black's model. The answer is slightly dependent on the strike price chosen. This procedure ignores any volatility smile in cap pricing.

(b) To calculate spot volatilities from flat volatilities the first step is usually to interpolate between the flat volatilities so that we have a flat volatility for each caplet payment date. We choose a strike price and use the flat volatilities to calculate cap prices. By subtracting successive cap prices we obtain caplet prices from which we can imply spot volatilities. The answer is slightly dependent on the strike price chosen. This procedure also ignores any volatility smile in caplet pricing.

**ASSIGNMENT QUESTIONS**

**Problem 28.21.**

Consider an eight-month European put option on a Treasury bond that currently has 14.25 years to maturity. The current cash bond price is $910, the exercise price is $900, and the volatility for the bond price is 10% per annum. A coupon of $35 will be paid by the bond in three months. The risk-free interest rate is 8% for all maturities up to one year. Use Black's model to determine the price of the option. Consider both the case where the
strike price corresponds to the cash price of the bond and the case where it corresponds to the quoted price.

The present value of the coupon payment is

\[ 35e^{-0.08 \times 0.25} = 34.31 \]

Equation (28.2) can therefore be used with \( F_B = (910 - 34.31)e^{0.08 \times 8/12} = 923.66, r = 0.08, \sigma_B = 0.10 \) and \( T = 0.6667 \). When the strike price is a cash price, \( K = 900 \) and

\[
\begin{align*}
    d_1 &= \frac{\ln(923.66/900) + 0.005 \times 0.6667}{0.1\sqrt{0.6667}} = 0.3587 \\
    d_2 &= d_1 - 0.1\sqrt{0.6667} = 0.2770
\end{align*}
\]

The option price is therefore

\[ 900e^{-0.08 \times 0.6667}N(-0.2770) - 875.69N(-0.3587) = 18.34 \]

or $18.34.

When the strike price is a quoted price 5 months of accrued interest must be added to 900 to get the cash strike price. The cash strike price is 900 + 35 \times 0.8333 = 929.17. In this case

\[
\begin{align*}
    d_1 &= \frac{\ln(923.66/929.17) + 0.005 \times 0.6667}{0.1\sqrt{0.6667}} = -0.0319 \\
    d_2 &= d_1 - 0.1\sqrt{0.6667} = -0.1136
\end{align*}
\]

and the option price is

\[ 929.17e^{-0.08 \times 0.6667}N(0.1136) - 875.69N(0.0319) = 31.22 \]

or $31.22.

**Problem 28.22.**

Calculate the price of a cap on the 90-day LIBOR rate in nine months' time when the principal amount is $1,000. Use Black's model and the following information:

a. The quoted nine-month Eurodollar futures price = 92. (Ignore differences between futures and forward rates.)

b. The interest-rate volatility implied by a nine-month Eurodollar option = 15% per annum.

c. The current 12-month interest rate with continuous compounding = 7.5% per annum.

d. The cap rate = 8% per annum. (Assume an actual/360 day count.)

The quoted futures price corresponds to a forward rate is 8% per annum with quarterly compounding and actual/360. The parameters for Black's model are therefore: \( F_k = 0.08, K = 0.08, R = 0.075, \sigma_k = 0.15, t_k = 0.75, \) and \( P(0, t_{k+1}) = e^{-0.075 \times 1} = 0.9277 \)

\[
\begin{align*}
    d_1 &= \frac{0.5 \times 0.15^2 \times 0.75}{0.15\sqrt{0.75}} = 0.0650 \\
    d_2 &= -\frac{0.5 \times 0.15^2 \times 0.75}{0.15\sqrt{0.75}} = -0.0650
\end{align*}
\]

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and the call price, \( c \), is given by
\[
c = 0.25 \times 1,000 \times 0.9277 \left[ 0.08N(0.0650) - 0.08N(-0.0650) \right] = 0.96
\]

**Problem 28.23.**

Suppose that the LIBOR yield curve is flat at 8% with annual compounding. A swaption gives the holder the right to receive 7.6% in a five-year swap starting in four years. Payments are made annually. The volatility of the forward swap rate is 25% per annum and the principal is $1 million. Use Black’s model to price the swaption. Compare your answer to that given by DerivaGem.

The payoff from the swaption is a series of five cash flows equal to \( \max[0.076 - s_T, 0] \) in million of dollars where \( s_T \) is the five-year swap rate in four years. The value of an annuity that provides $1 per year at the end of years 5, 6, 7, 8, and 9 is
\[
\sum_{t=5}^{9} \frac{1}{1.08^t} = 2.9348
\]

The value of the swaption in millions of dollars is therefore
\[
2.9348[0.076N(-d_2) - 0.08N(-d_1)]
\]

where
\[
d_1 = \frac{\ln(0.08/0.076) + 0.25^2 \times 4/2}{0.25\sqrt{4}} = 0.3526
\]
and
\[
d_2 = \frac{\ln(0.08/0.076) - 0.25^2 \times 4/2}{0.25\sqrt{4}} = -0.1474
\]

The value of the swaption is
\[
2.9348[0.076N(0.1474) - 0.08N(-0.3526)] = 0.039554
\]
or $39,554. This is the same answer as that given by DerivaGem. Note that for the purposes of using DerivaGem the zero rate is 7.696% continuously compounded for all maturities.

**Problem 28.24.**

Use the DerivaGem software to value a five-year collar that guarantees that the maximum and minimum interest rates on a LIBOR-based loan (with quarterly resets) are 5% and 7% respectively. The LIBOR zero curve (continuously compounded) is currently flat at 6%. Use a flat volatility of 20%. Assume that the principal is $100.

We use the Caps and Swap Options worksheet of DerivaGem. To set the zero curve as flat at 6% with continuous compounding, we need only enter 6% for one maturity. To value
the cap we select Cap/Floor as the Underlying Type, enter Quarterly for the Settlement Frequency, 100 for the Principal, 0 for the Start (Years), 5 for the End (Years), 7% for the Cap/Floor Rate, and 20% for the Volatility. We select Black-European as the Pricing Model and choose the Cap button. We do not check the Imply Breakeven Rate and Imply Volatility boxes. The value of the cap is 1.565. To value the floor we change the Cap/Floor Rate to 5% and select the Floor button rather than the Cap button. The value is 1.072. The collar is a long position in the cap and a short position in the floor. The value of the collar is therefore

\[ 1.565 - 1.072 = 0.493 \]

**Problem 28.25.**

*Use the DerivaGem software to value a European swap option that gives you the right in two years to enter into a 5-year swap in which you pay a fixed rate of 6% and receive floating. Cash flows are exchanged semiannually on the swap. The 1-year, 2-year, 5-year, and 10-year zero-coupon interest rates (continuously compounded) are 5%, 6%, 6.5%, and 7%, respectively. Assume a principal of $100 and a volatility of 15% per annum. Give an example of how the swap option might be used by a corporation. What bond option is equivalent to the swap option?*

We choose the third worksheet of DerivaGem and choose Swap Option as the Underlying Type. We enter 100 as the Principal, 2 as the Start (Years), 7 as the End (Years), 6% as the Swap Rate, and Semiannual as the Settlement Frequency. We also enter the zero curve information. We choose Black-European as the pricing model, enter 15% as the Volatility and check the Pay Fixed button. We do not check the Imply Breakeven Rate and Imply Volatility boxes. The value of the swap option is 4.606. For a company that expects to borrow at LIBOR plus 50 basis points in two years and then enter into a swap to convert to five-year fixed-rate borrowings, the swap guarantees that its effective fixed rate will not be more than 6.5%. The swap option is the same as an option to sell a five-year 6% coupon bond for par in two years.
CHAPTER 29
Convexity, Timing, and Quanto Adjustments

Notes for the Instructor

This chapter uses the results in Chapter 27. It starts by pointing out that when the expected value of a bond price is the forward price the expected value of a bond yield is not the forward yield. To value a product that provides a payoff at time $T$ dependent on a bond yield observed at that time we want in work in a world where the bond price equals its forward price. (This is a world that is forward risk neutral with respect to a zero-coupon bond maturing at time $T$.) We must therefore make an adjustment to the forward yield. This is referred to as a convexity adjustment. I like to go through Examples 29.1 and 29.2 fairly carefully to make sure students understand what is going on.

Timing adjustments and quanto adjustments are applications of the change of numeraire argument in Section 27.8. Again, when teaching this material, I make heavy use of the examples in the text (29.3, 29.4, and 29.5). I find Siegel's paradox works well as an application of the ideas in Section 29.3.

Problems 29.11, 29.12, and 29.13 are fairly straightforward applications of the material in the chapter. Problem 29.10 is a little more difficult. It can work well as an assignment or for class discussion.

QUESTIONS AND PROBLEMS

Problem 29.1.

Explain how you would value a derivative that pays off $100R$ in five years where $R$ is the one-year interest rate (annually compounded) observed in four years. What difference would it make if the payoff were in four years? What difference would it make if the payoff were in six years?

The value of the derivative is $100R_{4,5}P(0,5)$ where $P(0,t)$ is the value of a $t$-year zero-coupon bond today and $R_{t_1,t_2}$ is the forward rate for the period between $t_1$ and $t_2$, expressed with annual compounding. If the payoff is made in four years the value is $100(R_{4,5} + c)P(0,4)$ where $c$ is the convexity adjustment given by equation (29.2). The formula for the convexity adjustment is:

$$c = \frac{4R_{4,5}^2\sigma_{4,5}^2}{(1 + R_{4,5})}$$

where $\sigma_{t_1,t_2}$ is the volatility of the forward rate between times $t_1$ and $t_2$. 

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The expression $100(R_{4,5} + c)$ is the expected payoff in a world that is forward risk neutral with respect to a zero-coupon bond maturing at time four years. If the payoff is made in six years, the value is from equation (29.4) given by

$$100(R_{4,5} + c)P(0,6)\exp\left[-\frac{4\rho \sigma_{4,5} \sigma_{4,6} R_{4,6} \times 2}{1 + R_{4,6}}\right]$$

where $\rho$ is the correlation between the (4,5) and (4,6) forward rates. As an approximation we can assume that $\rho = 1$, $\sigma_{4,5} = \sigma_{4,6}$, and $R_{4,5} = R_{4,6}$. Approximating the exponential function we then get the value of the derivative as $100(R_{4,5} - c)P(0,6)$.

**Problem 29.2.**

Explain whether any convexity or timing adjustments are necessary when

a. We wish to value a spread option that pays off every quarter the excess (if any) of the five-year swap rate over the three-month LIBOR rate applied to a principal of $100. The payoff occurs 90 days after the rates are observed.

b. We wish to value a derivative that pays off every quarter the three-month LIBOR rate minus the three-month Treasury bill rate. The payoff occurs 90 days after the rates are observed.

(a) A convexity adjustment is necessary for the swap rate
(b) No convexity or timing adjustments are necessary.

**Problem 29.3.**

Suppose that in Example 28.3 of Section 28.2 the payoff occurs after one year (i.e., when the interest rate is observed) rather than in 15 months. What difference does this make to the inputs to Black’s models?

There are two differences. The discounting is done over a 1.0-year period instead of over a 1.25-year period. Also a convexity adjustment to the forward rate is necessary. From equation (29.2) the convexity adjustment is:

$$\frac{0.07^2 \times 0.2^2 \times 0.25 \times 1}{1 + 0.25 \times 0.07} = 0.00005$$

or about half a basis point.

In the formula for the caplet we set $F_k = 0.07005$ instead of 0.07. This means that $d_1 = -0.5642$ and $d_2 = -0.7642$. With continuous compounding the 15-month rate is 6.5% and the forward rate between 12 and 15 months is 6.94%. The 12 month rate is therefore 6.39% The caplet price becomes

$$0.25 \times 10,000e^{-0.069394 \times 1} \left[0.07005N(-0.5642) - 0.08N(-0.7642)\right] = 5.29$$

or $5.29.
Problem 29.4.

The yield curve is flat at 10% per annum with annual compounding. Calculate the value of an instrument where, in five years' time, the two-year swap rate (with annual compounding) is received and a fixed rate of 10% is paid. Both are applied to a notional principal of $100. Assume that the volatility of the swap rate is 20% per annum. Explain why the value of the instrument is different from zero.

The convexity adjustment discussed in Section 29.1 leads to the instrument being worth an amount slightly different from zero. Define $G(y)$ as the value as seen in five years of a two-year bond with a coupon of 10% as a function of its yield.

$$G(y) = \frac{0.1}{1+y} + \frac{1.1}{(1+y)^2}$$

$$G'(y) = -\frac{0.1}{(1+y)^2} - \frac{2.2}{(1+y)^3}$$

$$G''(y) = \frac{0.2}{(1+y)^3} + \frac{6.6}{(1+y)^4}$$

It follows that $G'(0.1) = -1.7355$ and $G''(0.1) = 4.6582$ and the convexity adjustment that must be made for the two-year swap rate is

$$0.5 \times 0.1^2 \times 0.2^2 \times 5 \times \frac{4.6582}{1.7355} = 0.00268$$

We can therefore value the instrument on the assumption that the swap rate will be 10.268% in five years. The value of the instrument is

$$\frac{0.268}{1.15} = 0.167$$

or $0.167.

Problem 29.5.

What difference does it make in Problem 29.4 if the swap rate is observed in five years, but the exchange of payments takes place in (a) six years, and (b) seven years? Assume that the volatilities of all forward rates are 20%. Assume also that the forward swap rate for the period between years five and seven has a correlation of 0.8 with the forward interest rate between years five and six and a correlation of 0.95 with the forward interest rate between years five and seven.

In this case we have to make a timing adjustment as well as a convexity adjustment to the forward swap rate. For (a) equation (29.4) shows that the timing adjustment involves multiplying the swap rate by

$$\exp \left[ -\frac{0.8 \times 0.20 \times 0.20 \times 0.1 \times 5}{1 + 0.1} \right] = 0.9856$$
so that it becomes $10.268 \times 0.9856 = 10.120$. The value of the instrument is

$$\frac{0.120}{1.16} = 0.068$$

or $\$0.068$.

For (b) equation (29.4) shows that the timing adjustment involves multiplying the swap rate by

$$\exp \left[ -\frac{0.95 \times 0.2 \times 0.2 \times 0.1 \times 2 \times 5}{1 + 0.1} \right] = 0.9660$$

so that it becomes $10.268 \times 0.966 = 9.919$. The value of the instrument is now

$$\frac{-0.081}{1.17} = -0.042$$

or $-\$0.042$.

**Problem 29.6.**

The price of a bond at time $T$, measured in terms of its yield, is $G(y_T)$. Assume geometric Brownian motion for the forward bond yield, $y$, in a world that is forward risk neutral with respect to a bond maturing at time $T$. Suppose that the growth rate of the forward bond yield is $\alpha$ and its volatility $\sigma_y$.

a. Use Itō’s lemma to calculate the process for the forward bond price in terms of $\alpha$, $\sigma_y$, $y$, and $G(y)$.

b. The forward bond price should follow a martingale in the world considered. Use this fact to calculate an expression for $\alpha$.

c. Show that the expression for $\alpha$ is, to a first approximation, consistent with equation (29.1).

(a) The process for $y$ is

$$dy = \alpha y dt + \sigma_y y dz$$

The forward bond price is $G(y)$. From Itō’s lemma, its process is

$$d[G(y)] = [G'(y)\alpha y + \frac{1}{2}G''(y)\sigma_y^2 y^2] dt + G'(y)\sigma_y y dz$$

(b) Since the expected growth rate of $G(y)$ is zero

$$G'(y)\alpha y + \frac{1}{2}G''(y)\sigma_y^2 y^2 = 0$$

or

$$\alpha = -\frac{1}{2} \frac{G''(y)}{G'(y)} \sigma_y^2 y$$

(c) Assuming as an approximation that $y$ always equals its initial value of $y_0$, this shows that the growth rate of $y$ is

$$\frac{-1}{2} \frac{G''(y_0)}{G'(y_0)} \sigma_y^2 y_0$$

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The variable $y$ starts at $y_0$ and ends as $y_T$. The convexity adjustment to $y_0$ when we are calculating the expected value of $y_T$ in a world that is forward risk neutral with respect to a zero-coupon bond maturing at time $T$ is approximately $y_0T$ times this or

$$\frac{1}{2} \frac{G''(y_0)}{G'(y_0)} y_0^2 y_0^2 T$$

This is consistent with equation (29.1).

**Problem 29.7.**

The variable $S$ is an investment asset providing income at rate $q$ measured in currency $A$. It follows the process

$$dS = \mu_S S \, dt + \sigma_S S \, dz$$

in the real world. Defining new variables as necessary, give the process followed by $S$, and the corresponding market price of risk, in

(a) A world that is the traditional risk-neutral world for currency $A$.

(b) A world that is the traditional risk-neutral world for currency $B$.

(c) A world that is forward risk neutral with respect to a zero-coupon currency $A$ bond maturing at time $T$.

(d) A world that is forward risk neutral with respect to a zero coupon currency $B$ bond maturing at time $T$.

(a) In the traditional risk-neutral world the process followed by $S$ is

$$dS = (r - q)S \, dt + \sigma_S S \, dz$$

where $r$ is the instantaneous risk-free rate. The market price of $dz$-risk is zero.

(b) In the traditional risk-neutral world for currency $B$ the process is

$$dS = (r - q + \rho_{QS} \sigma_S \sigma_Q)S \, dt + \sigma_S S \, dz$$

where $Q$ is the exchange rate (units of $A$ per unit of $B$), $\sigma_Q$ is the volatility of $Q$ and $\rho_{QS}$ is the coefficient of correlation between $Q$ and $S$. The market price of $dz$-risk is $\rho_{QS} \sigma_Q$

(c) In a world that is forward risk neutral with respect to a zero-coupon bond in currency $A$ maturing at time $T$

$$dS = (r - q + \sigma_S \sigma_P)S \, dt + \sigma_S S \, dz$$

where $\sigma_P$ is the bond price volatility. The market price of $dz$-risk is $\sigma_P$

(d) In a world that is forward risk neutral with respect to a zero-coupon bond in currency $B$ maturing at time $T$

$$dS = (r - q + \sigma_S \sigma_P + \rho_{FS} \sigma_S \sigma_F)S \, dt + \sigma_S S \, dz$$
where \( F \) is the forward exchange rate, \( \sigma_F \) is the volatility of \( F \) (units of A per unit of B), and \( \rho_{FS} \) is the correlation between \( F \) and \( S \). The market price of \( dz \)-risk is \( \sigma_F + \rho_{FS} \sigma_F \).

**Problem 29.8.**

A call option provides a payoff at time \( T \) of \( \max(S_T - K, 0) \) yen, where \( S_T \) is the dollar price of gold at time \( T \) and \( K \) is the strike price. Assuming that the storage costs of gold are zero and defining other variables as necessary, calculate the value of the contract.

Define

\[
P(t,T) : \text{Price in yen at time } t \text{ of a bond paying 1 yen at time } T
\]

\[
E_T(\cdot) : \text{Expectation in world that is forward risk neutral with respect to } P(t,T)
\]

\[
F : \text{Dollar forward price of gold for a contract maturing at time } T
\]

\[
F_0 : \text{Value of } F \text{ at time zero}
\]

\[
\sigma_F : \text{Volatility of } F
\]

\[
G : \text{Forward exchange rate (dollars per yen)}
\]

\[
\sigma_G : \text{Volatility of } G
\]

We assume that \( S_T \) is lognormal. We can work in a world that is forward risk neutral with respect to \( P(t,T) \) to get the value of the call as

\[
P(0,T)[E_T(S_T)N(d_1) - N(d_2)]
\]

where

\[
d_1 = \frac{\ln[E_T(S_T)/K] + \sigma_F^2 T/2}{\sigma_F \sqrt{T}}
\]

\[
d_2 = \frac{\ln[E_T(S_T)/K] - \sigma_F^2 T/2}{\sigma_F \sqrt{T}}
\]

The expected gold price in a world that is forward risk-neutral with respect to a zero-coupon dollar bond maturing at time \( T \) is \( F_0 \). It follows from equation (29.6) that

\[
E_T(S_T) = F_0 (1 + \rho \sigma_F \sigma_G T)
\]

Hence the option price, measured in yen, is

\[
P(0,T)[F_0 (1 + \rho \sigma_F \sigma_G T) N(d_1) - K N(d_2)]
\]

where

\[
d_1 = \frac{\ln[F_0 (1 + \rho \sigma_F \sigma_G T)/K] + \sigma_F^2 T/2}{\sigma_F \sqrt{T}}
\]

\[
d_2 = \frac{\ln[F_0 (1 + \rho \sigma_F \sigma_G T)/K] - \sigma_F^2 T/2}{\sigma_F \sqrt{T}}
\]
Problem 29.9.

Suppose that an index of Canadian stocks currently stands at 400. The Canadian dollar is currently worth 0.70 U.S. dollars. The risk-free interest rates in Canada and the U.S. are constant at 6% and 4%, respectively. The dividend yield on the index is 3%. Define $Q$ as the number of Canadian dollars per U.S. dollar and $S$ as the value of the index. The volatility of $S$ is 20%, the volatility of $Q$ is 6%, and the correlation between $S$ and $Q$ is 0.4. Use DerivaGem to determine the value of a two year American-style call option on the index if

(a) It pays off in Canadian dollars the amount by which the index exceeds 400.
(b) It pays off in U.S. dollars the amount by which the index exceeds 400.

(a) The value of the option can be calculated by setting $S_0 = 400$, $K = 400$, $r = 0.06$, $q = 0.03$, $\sigma = 0.2$, and $T = 2$. With 100 time steps the value (in Canadian dollars) is 52.92.

(b) The growth rate of the index using the CDN numeraire is $0.06 - 0.03 = 3\%$. When we switch to the USD numeraire we increase the growth rate of the index by $0.4 \times 0.2 \times 0.06$ or $0.48\%$ per year to $3.48\%$. The option can therefore be calculated using DerivaGem with $S_0 = 400$, $K = 400$, $r = 0.04$, $q = 0.04 - 0.0348 = 0.0052$, $\sigma = 0.2$, and $T = 2$. With 100 time steps DerivaGem gives the value as 57.51.

ASSIGNMENT QUESTIONS

Problem 29.10.

Consider an instrument that will pay off $S$ dollars in two years where $S$ is the value of the Nikkei index. The index is currently 20,000. The dollar–yen exchange rate (yen per dollar) is 100. The correlation between the exchange rate and the index is 0.3 and the dividend yield on the index is 1% per annum. The volatility of the Nikkei index is 20% and the volatility of the yen–dollar exchange rate is 12%. The interest rates (assumed constant) in the U.S. and Japan are 4% and 2%, respectively.

(a) What is the value of the instrument
(b) Suppose that the exchange rate at some point during the life of the instrument is $Q$ and the level of the index is $S$. Show that a U.S. investor can create a portfolio that changes in value by approximately $\Delta S$ dollar when the index changes in value by $\Delta S$ yen by investing $S$ dollars in the Nikkei and shorting $SQ$ yen.
(c) Confirm that this is correct by supposing that the index changes from 20,000 to 20,050 and the exchange rate changes from 100 to 99.7.
(d) How would you delta hedge the instrument under consideration?

(a) We require the expected value of the Nikkei index in a dollar risk-neutral world. In a yen risk-neutral world the expected value of the index is $20,000e^{(0.02-0.01)x2} = 20,404.03$. In a dollar risk-neutral world the analysis in Section 29.3 shows that this becomes

$$20,404.03e^{0.3\times0.20\times0.12\times2} = 20,699.97$$
The value of the instrument is therefore

\[ 20,699.97e^{-0.04\times2} = 19,108.48 \]

(b) An amount \( SQ \) yen is invested in the Nikkei. Its value in yen changes to

\[ SQ \left( 1 + \frac{\Delta S}{S} \right) \]

In dollars this is worth

\[ SQ \frac{1 + \Delta S/S}{Q + \Delta Q} \]

where \( \Delta Q \) is the increase in \( Q \). When terms of order two and higher are ignored, the dollar value becomes

\[ S(1 + \Delta S/S - \Delta Q/Q) \]

The gain on the Nikkei position is therefore \( \Delta S - S\Delta Q/Q \)

When \( SQ \) yen are shorted the gain in dollars is

\[ SQ \left( \frac{1}{Q} - \frac{1}{Q + \Delta Q} \right) \]

This equals \( S\Delta Q/Q \) when terms of order two and higher are ignored. The gain on the whole position is therefore \( \Delta S \) as required.

(c) In this case the investor invests \$20,000 in the Nikkei. The investor converts the funds to yen and buys 100 times the index. The index rises to 20,050 so that the investment becomes worth 2,005,000 yen or

\[ \frac{2,005,000}{99.7} = 20,110.33 \]

dollars. The investor therefore gains \$110.33. The investor also shorts 2,000,000 yen. The value of the yen changes from \$0.0100 to \$0.01003. The investor therefore loses \( 0.00003 \times 2,000,000 = 60 \) dollars on the short position. The net gain is 50.33 dollars. This is close to the required gain of \$50.

(d) Suppose that the value of the instrument is \( V \). When the index changes by \( \Delta S \) yen the value of the instrument changes by

\[ \frac{\partial V}{\partial S} \Delta S \]

dollars. We can calculate \( \partial V/\partial S \). Part (b) of this question shows how to manufacture an instrument that changes by \( \Delta S \) dollars. This enables us to delta-hedge our exposure to the index.
Problem 29.11.
Suppose that the LIBOR yield curve is flat at 8% (with continuous compounding). The payoff from a derivative occurs in four years. It is equal to the five-year rate minus the two-year rate at this time, applied to a principal of $100 with both rates being continuously compounded. (The payoff can be positive or negative.) Calculate the value of the derivative. Assume that the volatility for all rates is 25%. What difference does it make if the payoff occurs in five years instead of four years? Assume all rates are perfectly correlated.

To calculate the convexity adjustment for the five-year rate define the price of a five year bond, as a function of its yield as

\[ G(y) = e^{-5y} \]

\[ G'(y) = -5e^{-5y} \]

\[ G''(y) = 25e^{-5y} \]

The convexity adjustment is

\[ 0.5 \times 0.08^2 \times 0.25^2 \times 4 \times 5 = 0.004 \]

Similarly for the two year rate the convexity adjustment is

\[ 0.5 \times 0.08^2 \times 0.25^2 \times 4 \times 2 = 0.0016 \]

We can therefore value the derivative by assuming that the five year rate is 8.4% and the two-year rate is 8.16%. The value of the derivative is

\[ 0.24e^{-0.08 \times 4} = 0.174 \]

If the payoff occurs in five years rather than four years it is necessary to make a timing adjustment. From equation (29.4) this involves multiplying the forward rate by

\[ \exp \left[ -\frac{1 \times 0.25 \times 0.25 \times 0.08 \times 4 \times 1}{1.08} \right] = 0.98165 \]

The value of the derivative is

\[ 0.24 \times 0.98165e^{-0.08 \times 5} = 0.158 \]

Problem 29.12.
Suppose that the payoff from a derivative will occur in ten years and will equal the three-year U.S. dollar swap rate for a semiannual-pay swap observed at that time applied to a certain principal. Assume that the yield curve is flat at 8% (semiannually compounded) per annum in dollars and 3% (semiannually compounded) in yen. The forward swap rate
volatility is 18%, the volatility of the ten year “yen per dollar” forward exchange rate is 12%, and the correlation between this exchange rate and U.S. dollar interest rates is 0.25.

a. What is the value of the derivative if the swap rate is applied to a principal of $100 million so that the payoff is in dollars?

b. What is its value of the derivative if the swap rate is applied to a principal of 100 million yen so that the payoff is in yen?

(a) In this case we must make a convexity adjustment to the forward swap rate. Define

\[ G(y) = \sum_{i=1}^{6} \frac{4}{(1+y/2)^i} + \frac{100}{(1+y/2)^6} \]

so that

\[ G'(y) = -\sum_{i=1}^{6} \frac{2i}{(1+y/2)^{i+1}} + \frac{300}{(1+y/2)^7} \]

\[ G''(y) = \sum_{i=1}^{6} \frac{i(i+1)}{(1+y/2)^{i+2}} + \frac{1050}{(1+y/2)^8} \]

\[ G'(0.08) = -262.11 \text{ and } G''(0.08) = 853.29 \text{ so that the convexity adjustment is } \]

\[ \frac{1}{2} \times 0.08^2 \times 0.18^2 \times 10 \times \frac{853.29}{262.11} = 0.00338 \]

The adjusted forward swap rate is 0.08 + 0.00338 = 0.08338 and the value of the derivative in millions of dollars is

\[ \frac{0.08338 \times 100}{1.04^{20}} = 3.805 \]

(b) When the swap rate is applied to a yen principal we must make a quanto adjustment in addition to the convexity adjustment. From Section 29.3 this involves multiplying the forward swap rate by \( e^{-0.25 \times 0.12 \times 0.18 \times 10} = 0.9474 \). (Note that the correlation is the correlation between the dollar per yen exchange rate and the swap rate. It is therefore -0.25 rather than +0.25.) The value of the derivative in millions of yen is

\[ \frac{0.08338 \times 0.9474 \times 100}{1.015^{20}} = 5.865 \]

Problem 29.13.

The payoff from a derivative will occur in 8 years. It will equal the average of the one-year interest rates observed at times 5, 6, 7, and 8 years applied to a principal of $1,000. The yield curve is flat at 6% with annual compounding and the volatilities of all rates are 16%. Assume perfect correlation between all rates. What is the value of the derivative?
No adjustment is necessary for the forward rate applying to the period between years seven and eight. Using this, we can deduce from equation (29.4) that the forward rate applying to the period between years five and six must be multiplied by

$$\exp \left[ -\frac{1 \times 0.16 \times 0.16 \times 0.06 \times 5 \times 2}{1.06} \right] = 0.9856$$

Similarly the forward rate applying to the period between year six and year seven must be multiplied by

$$\exp \left[ -\frac{1 \times 0.16 \times 0.16 \times 0.06 \times 6 \times 1}{1.06} \right] = 0.9913$$

Similarly the forward rate applying to the period between year eight and nine must be multiplied by

$$\exp \left[ -\frac{1 \times 0.16 \times 0.16 \times 0.06 \times 8 \times 1}{1.06} \right] = 1.0117$$

The adjusted forward average interest rate is therefore

$$0.25 \times (0.08 \times 0.9856 + 0.08 \times 0.9913 + 0.08 + 0.08 \times 1.0117) = 0.07977$$

The value of the derivative is

$$0.7977 \times 1000 \times 1.06^{-8} = 50.05$$
CHAPTER 30
Interest Rate Derivatives: Models of the Short Rate

Notes for the Instructor

This chapter covers equilibrium and no-arbitrage models of the short rate. The first part of the chapter explains that if we know all the details of the behavior of the short-term interest rate in a risk-neutral world, we can deduce the behavior of the whole term structure. It reviews the well known equilibrium models such as Vasicek and CIR.

The second part of the chapter on no-arbitrage models explains how one-factor models of the short rate can be constructed so that they are exactly consistent with the zero rates observed in the market today. These types of models are popular with practitioners. The chapter describes a trinomial tree-building procedures that can be used to represent a wide range of different models. DerivaGem can be used to display trees in class as this material is taught.

Problems 30.22 to 30.26 can be used as assignment questions. All make use of DerivaGem.

QUESTIONS AND PROBLEMS

Problem 30.1.
What is the difference between an equilibrium model and a no-arbitrage model?

Equilibrium models usually start with assumptions about economic variables and derive the behavior of interest rates. The initial term structure is an output from the model. In a no-arbitrage model the initial term structure is an input. The behavior of interest rates in a no-arbitrage model is designed to be consistent with the initial term structure.

Problem 30.2.
Suppose that the short rate is currently 4% and its standard deviation is 1% per annum. What happens to the standard deviation when the short rate increases to 8% in (a) Vasicek's model; (b) Rendleman and Bartter's model; and (c) the Cox, Ingersoll, and Ross model?

In Vasicek's model the standard deviation stays at 1%. In the Rendleman and Bartter model the standard deviation is proportional to the level of the short rate. When the short rate increases from 4% to 8% the standard deviation increases from 1% to 2%. In the Cox, Ingersoll, and Ross model the standard deviation of the short rate is proportional to the square root of the short rate. When the short rate increases from 4% to 8% the standard deviation of the short rate increases from 1% to 1.414%.
Problem 30.3.

If a stock price were mean reverting or followed a path-dependent process there would be market inefficiency. Why is there not a market inefficiency when the short-term interest rate does so?

If the price of a traded security followed a mean-reverting or path-dependent process there would be a market inefficiency. The short-term interest rate is not the price of a traded security. In other words we cannot trade something whose price is always the short-term interest rate. There is therefore no market inefficiency when the short-term interest rate follows a mean-reverting or path-dependent process. We can trade bonds and other instruments whose prices do depend on the short rate. The prices of these instruments do not follow mean-reverting or path-dependent processes.

Problem 30.4.

Explain the difference between a one-factor and a two-factor interest rate model.

In a one-factor model there is one source of uncertainty driving all rates. This usually means that in any short period of time all rates move in the same direction (but not necessarily by the same amount). In a two-factor model, there are two sources of uncertainty driving all rates. The first source of uncertainty usually gives rise to a roughly parallel shift in rates. The second gives rise to a twist where long and short rates moves in opposite directions.

Problem 30.5.

Can the approach described in Section 30.4 for decomposing an option on a coupon-bearing bond into a portfolio of options on zero-coupon bonds be used in conjunction with a two-factor model? Explain your answer.

No. The approach in Section 30.4 relies on the argument that, at any given time, all bond prices are moving in the same direction. This is not true when there is more than one factor.

Problem 30.6.

Suppose that $a = 0.1$ and $b = 0.1$ in both the Vasicek and the Cox, Ingersoll, Ross model. In both models, the initial short rate is 10% and the initial standard deviation of the short rate change in a short time $\Delta t$ is $0.02\sqrt{\Delta t}$. Compare the prices given by the models for a zero-coupon bond that matures in year 10.

In Vasicek's model, $a = 0.1$, $b = 0.1$, and $\sigma = 0.02$ so that

$$B(t, t + 10) = \frac{1}{0.1} (1 - e^{-0.1 \times 10}) = 6.32121$$

$$A(t, t + 10) = \exp \left[ \frac{(6.32121 - 10)(0.1^2 \times 0.1 - 0.0002)}{0.01} \cdot \frac{0.0004 \times 6.32121^2}{0.4} \right]$$

$$= 0.71587$$

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The bond price is therefore \( \frac{0.71587e^{-6.32121 \times 0.1}}{0.38046} = 0.38046 \)

In the Cox, Ingersoll, and Ross model, \( a = 0.1, b = 0.1 \) and \( \sigma = 0.02/\sqrt{0.1} = 0.0632 \).

Also

\[
\gamma = \sqrt{a^2 + 2\sigma^2} = 0.13416
\]

Define

\[
\beta = (\gamma + a)(e^{10\gamma} - 1) + 2\gamma = 0.92992
\]

\[
B(t, t + 10) = \frac{2(e^{10\gamma} - 1)}{\beta} = 6.07650
\]

\[
A(t, t + 10) = \left( \frac{2\gamma e^{5(a+\gamma)}}{\beta} \right)^{2ab/\sigma^2} = 0.69746
\]

The bond price is therefore \( \frac{0.69746e^{-6.07650 \times 0.1}}{0.37986} = 0.37986 \)

**Problem 30.7.**

Suppose that \( a = 0.1, b = 0.08, \) and \( \sigma = 0.015 \) in Vasicek’s model with the initial value of the short rate being 5%. Calculate the price of a one-year European call option on a zero-coupon bond with a principal of $100 that matures in three years when the strike price is $87.

Using the notation in the text, \( s = 3, T = 1, L = 100, K = 87, \) and

\[
\sigma_P = \frac{0.015}{0.1} (1 - e^{-2 \times 0.1}) \sqrt{\frac{1 - e^{-2 \times 0.1 \times 1}}{2 \times 0.1}} = 0.025886
\]

From equation (30.6), \( P(0, 1) = 0.94988, P(0, 3) = 0.85092, \) and \( h = 1.14277 \) so that equation (30.20) gives the call price as call price is

\[
100 \times 0.85092 \times N(1.14277) - 87 \times 0.94988 \times N(1.11688) = 2.59
\]

or $2.59.

**Problem 30.8.**

Repeat Problem 30.7 valuing a European put option with a strike of $87. What is the put–call parity relationship between the prices of European call and put options? Show that the put and call option prices satisfy put–call parity in this case.

As mentioned in the text, equation (30.20) for a call option is essentially the same as Black’s model. There is a typo in the first printing of the book. By analogy with Black’s formulas corresponding expression for a put option is

\[
KP(0, T)N(-h + \sigma_P) - LP(0, s)N(-h)
\]

In this case the put price is

\[
87 \times 0.94988 \times N(-1.11688) - 100 \times 0.85092 \times N(-1.14277) = 0.14
\]
Since the underlying bond pays no coupon, put–call parity states that the put price plus the bond price should equal the call price plus the present value of the strike price. The bond price is 85.09 and the present value of the strike price is \(87 \times 0.94988 = 82.64\). Put–call parity is therefore satisfied:

\[
82.64 + 2.59 = 85.09 + 0.14
\]

**Problem 30.9.**

Suppose that \(a = 0.05\), \(b = 0.08\), and \(\sigma = 0.015\) in Vasicek's model with the initial short-term interest rate being 6%. Calculate the price of a 2.1-year European call option on a bond that will mature in three years. Suppose that the bond pays a coupon of 5% semiannually. The principal of the bond is 100 and the strike price of the option is 99. The strike price is the cash price (not the quoted price) that will be paid for the bond.

As explained in Section 30.4, the first stage is to calculate the value of \(r\) at time 2.1 years which is such that the value of the bond at that time is 99. Denoting this value of \(r\) by \(r^*\), we must solve

\[
2.5A(2.1, 2.5)e^{-B(2.1, 2.5)r^*} + 102.5A(2.1, 3.0)e^{-B(2.1, 3.0)r^*} = 99
\]

where the \(A\) and \(B\) functions are given by equations (30.7) and (30.8). The solution to this is \(r^* = 0.066\). Since

\[
2.5A(2.1, 2.5)e^{-B(2.1, 2.5)\times 0.066} = 2.43473
\]

and

\[
102.5A(2.1, 3.0)e^{-B(2.1, 3.0)\times 0.063} = 96.56438
\]

the call option on the coupon-bearing bond can be decomposed into a call option with a strike price of 2.43473 on a bond that pays off 2.5 at time 2.5 years and a call option with a strike price of 96.56438 on a bond that pays off 102.5 at time 3.0 years. Equation (30.20) shows that the value of the first option is 0.009085 and the value of the second option is 0.806143. The total value of the option is therefore 0.815238.

**Problem 30.10.**

Use the answer to Problem 30.9 and put–call parity arguments to calculate the price of a put option that has the same terms as the call option in Problem 30.9.

Put-call parity shows that:

\[
c + I + PV(K) = p + B_0
\]

or

\[
p = c + PV(K) - (B_0 - I)
\]

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where $c$ is the call price, $K$ is the strike price, $I$ is the present value of the coupons, and $B_0$ is the bond price. In this case $c = 0.8152$, $PV(K) = 99 \times P(0, 2.1) = 87.1222$, $B_0 - I = 2.5 \times P(0, 2.5) + 102.5 \times P(0, 3) = 87.4730$ so that the put price is

$$0.8152 + 87.1222 - 87.4730 = 0.4644$$

**Problem 30.11.**

In the Hull–White model, $a = 0.08$ and $\sigma = 0.01$. Calculate the price of a one-year European call option on a zero-coupon bond that will mature in five years when the term structure is flat at 10%, the principal of the bond is $100$, and the strike price is $68$.

Using the notation in the text $P(0, T) = e^{-0.1 \times 1} = 0.9048$ and $P(0, s) = e^{-0.1 \times 5} = 0.6065$. Also

$$\sigma_p = \frac{0.01}{0.08} (1 - e^{-4 \times 0.08}) \sqrt{\frac{1 - e^{-2 \times 0.08 \times 1}}{2 \times 0.08}} = 0.0329$$

and $h = -0.4192$ so that the call price is

$$100 \times 0.6065 N(h) - 68 \times 0.9048 N(h - \sigma_p) = 0.439$$

**Problem 30.12.**

Suppose that $a = 0.05$ and $\sigma = 0.015$ in the Hull–White model with the initial term structure being flat at 6% with semiannual compounding. Calculate the price of a 2.1-year European call option on a bond that will mature in three years. Suppose that the bond pays a coupon of 5% per annum semiannually. The principal of the bond is 100 and the strike price of the option is 99. The strike price is the cash price (not the quoted price) that will be paid for the bond.

The relevant parameters for the Hull–White model are $a = 0.05$ and $\sigma = 0.015$. Setting $\Delta t = 0.4$

$$\hat{B}(2.1, 3) = \frac{B(2.1, 3)}{B(2.1, 2.5)} \times 0.4 = 0.88888$$

Also from equation (30.26), $\hat{A}(2.1, 3) = 0.99925$ The first stage is to calculate the value of $R$ at time 2.1 years which is such that the value of the bond at that time is 99. Denoting this value of $R$ by $R^*$, we must solve

$$2.5 e^{-R^* \times 0.4} + 102.5 \hat{A}(2.1, 3) e^{-\hat{B}(2.1, 3) R^*} = 99$$

The solution to this for $R^*$ turns out to be 6.626%. The option on the coupon bond is decomposed into an option with a strike price of 96.565 on a zero-coupon bond with a principal of 102.5 and an option with a strike price of 2.435 on a zero-coupon bond with a principal of 2.5. The first option is worth $0.0103$ and the second option is worth $0.9343$. The total value of the option is therefore $0.9446$. (Note that the initial short rate with continuous compounding is 5.91%).
Problem 30.13.

Use a change of numeraire argument to show that the relationship between the futures rate and forward rate for the Ho–Lee model is as shown in Section 6.4. Use the relationship to verify the expression for $\theta(t)$ given for the Ho–Lee model in equation (30.11) (Hint The futures price is a martingale when the market price of risk is zero. The forward price is a martingale when the market price of risk is a zero-coupon bond maturing at the same time as the forward contract.)

We will consider instantaneous forward and futures rates. (A more general result involving the forward and futures rate applying to a period of time between $T_1$ and $T_2$ is proved in Technical Note 1 on the author’s site.)

Because $P(t, T) = A(t, T)e^{-r(T-t)}$ the process for $P(t, T)$ is from Itô’s lemma

$$dP(t, T) = \ldots - \sigma(T-t)P(t, T)dz$$

Define $F(t, T)$ as the instantaneous forward rate for maturity $T$. The process for $F(0, T)$ is from Itô’s lemma

$$dF(0, T) = \ldots + \sigma dz$$

The instantaneous forward rate with maturity $T$ has a drift of zero in a world that is forward risk neutral with respect to $P(t, T)$. This is a world where the market price of risk is $-\sigma(T-t)$. When we move to a world where the market price of risk is zero the drift of the forward rate increases to $\sigma^2(T-t)$. Integrating this between $t = 0$ and $t = T$ we see that the forward rate grows by a total of $\sigma^2 T^2/2$ between time 0 and time $T$ in a world where the market price of risk is zero. The futures rate has zero growth rate in this world. At time $T$ the forward rate equals the futures rate. It follows that the futures rate must exceed the forward rate by $\sigma^2 T^2/2$ at time zero. This is consistent with the formula in Section 6.4.

Define $G(0, t)$ as the instantaneous futures rate for maturity $t$ so that

$$G(0, t) - F(0, t) = \sigma^2 t^2/2$$

and

$$G_t(0, t) - F_t(0, t) = \sigma^2 t$$

In the traditional risk-neutral world the expected value of $r$ at time $t$ is the futures rate, $G(0, t)$. This means that the expected growth in $r$ at time $t$ must be $G_t(0, t)$ so that $\theta(t) = G_t(0, t)$. It follows that

$$\theta(t) = F_t(0, t) + \sigma^2 t$$

This is equation (30.11).

Problem 30.14.

Use a similar approach to that in Problem 30.13 to derive the relationship between the futures rate and the forward rate for the Hull–White model. Use the relationship to verify the expression for $\theta(t)$ given for the Hull–White model in equation (30.14)
In this case we have \( P(t, T) = A(t, T)e^{-B(t, T)r} \) so that from Itô's lemma
\[
dP(t, T) = \ldots - \sigma B(t, T)P(t, T)dz
\]

Define \( F(t, T) \) as the instantaneous forward rate for maturity \( T \). The process for \( F(0, T) \) is from Itô's lemma
\[
dF(0, T) = \ldots + \sigma e^{-a(T-t)}dZ
\]

This has drift of zero in a world that is forward risk neutral with respect to \( P(t, T) \). This is a world where the market price of risk is \(-\sigma B(t, T)\). When we move to a world where the market price of risk is zero the drift of \( F(0, T) \) increases to \( \sigma^2 e^{-a(T-t)}B(t, T) \). Integrating this between \( t = 0 \) and \( t = T \) we see that the forward rate grows by a total of
\[
\frac{\sigma^2}{2a^2}(1 - e^{-aT})^2
\]

between time 0 and time \( T \) in a world where the market price of risk is zero. The futures price has zero growth rate in this world. At time \( T \) the forward price equals the futures price. It follows that the futures price must exceed the forward price by
\[
\frac{\sigma^2}{2a^2}(1 - e^{-aT})^2
\]
at time zero.\(^3\)

\(^3\) To produce a result relating the futures rate for the period between times \( T_1 \) and \( T_2 \) to the forward rate between this period we can proceed as in Technical Note 1 on the author's web site. The drift of the forward rate is
\[
\frac{\sigma^2 B(t, T_2)^2 - \sigma^2 B(t, T_1)^2}{2(T_2 - T_1)}
\]
\[
= \frac{\sigma^2}{2a^2(T_2 - T_1)}[e^{at}(-2e^{-aT_2} + 2e^{-aT_1}) + e^{2at}[e^{-2aT_2} - e^{-2aT_1}]
\]

Integrating between time 0 and time \( T_1 \) we get
\[
\frac{\sigma^2}{2a^2(T_2 - T_1)}[(e^{aT_1} - 1)(-2e^{-aT_2} + 2e^{-aT_1})/a + (e^{2aT_1} - 1)(e^{-2aT_2} - e^{-2aT_1})/(2a)]
\]
\[
= \frac{\sigma^2 B(T_1, T_2)}{4a^2(T_2 - T_1)}[4(1 - e^{-aT_1}) - (1 - e^{-2aT_1})(1 + e^{-a(T_2 - T_1)})]
\]
\[
= \frac{B(T_1, T_2)}{T_2 - T_1}[B(T_1, T_2)(1 - e^{-2aT_1}) + 2aB(0, T_1)^2\frac{\sigma^2}{4a}]
\]

This is the amount by which the futures rate exceeds the forward rate at time zero.
Define \( G(0,t) \) as the instantaneous futures rate for maturity \( t \) so that

\[
G(0,t) - F(0,t) = \frac{\sigma^2}{2\alpha^2} (1 - e^{-at})^2
\]

and

\[
G_t(0,t) - F_t(0,t) = \frac{\sigma^2}{\alpha} (1 - e^{-at}) e^{-at}
\]

In the traditional risk-neutral world the expected value of \( r \) at time \( t \) is the futures rate, \( G(0,t) \). This means that the expected growth in \( r \) at time \( t \) must be \( G_t(0,t) - a[r - G(0,t)] \) so that \( \theta(t) = G_t(0,t) - a[r - G(0,t)] \). It follows that

\[
\theta(t) = G_t(0,t) + aG(0,t)
\]

\[
= F_t(0,t) + aF(0,t) + \frac{\sigma^2}{\alpha} (1 - e^{-at}) e^{-at} + \frac{\sigma^2}{2\alpha} (1 - e^{-at})^2
\]

\[
= F_t(0,t) + aF(0,t) + \frac{\sigma^2}{2\alpha} (1 - e^{-2at})
\]

This proves equation (30.14).

**Problem 30.15.**

Suppose that \( a = 0.05, \sigma = 0.015 \), and the term structure is flat at 10%. Construct a trinomial tree for the Hull–White model where there are two time steps, each one year in length.

The time step, \( \Delta t \), is 1 so that \( \Delta r = 0.015\sqrt{3} = 0.02598 \). Also \( j_{\text{max}} = 4 \) showing that the branching method should change four steps from the center of the tree. With only three steps we never reach the point where the branching changes. The tree is shown in Figure S30.1.

**Problem 30.16.**

Calculate the price of a two-year zero-coupon bond from the tree in Figure 30.6.

A two-year zero-coupon bond pays off $100 at the ends of the final branches. At node B it is worth \( 100e^{-0.12 \times 1} = 88.69 \). At node C it is worth \( 100e^{-0.10 \times 1} = 90.48 \). At node D it is worth \( 100e^{-0.08 \times 1} = 92.31 \). It follows that at node A the bond is worth

\[
(88.69 \times 0.25 + 90.48 \times 0.5 + 92.31 \times 0.25)e^{-0.1 \times 1} = 81.88
\]

or $81.88

**Problem 30.17.**

Calculate the price of a two-year zero-coupon bond from the tree in Figure 30.9 and verify that it agrees with the initial term structure.
A two-year zero-coupon bond pays off $100 at time two years. At node B it is worth $100e^{-0.00693 \times 1} = 93.30$. At node C it is worth $100e^{-0.00520 \times 1} = 94.93$. At node D it is worth $100e^{-0.00347 \times 1} = 96.59$. It follows that at node A the bond is worth $(93.30 \times 0.167 + 94.93 \times 0.666 + 96.59 \times 0.167)e^{-0.0382 \times 1} = 91.37$ or $91.37$. Because $91.37 = 100e^{-0.04512 \times 2}$, the price of the two-year bond agrees with the initial term structure.

**Problem 30.18.**

Calculate the price of an 18-month zero-coupon bond from the tree in Figure 30.10 and verify that it agrees with the initial term structure.

An 18-month zero-coupon bond pays off $100 at the final nodes of the tree. At node E it is worth $100e^{-0.068 \times 0.5} = 95.70$. At node F it is worth $100e^{-0.0648 \times 0.5} = 96.81$. At node G it is worth $100e^{-0.0477 \times 0.5} = 97.64$. At node H it is worth $100e^{-0.0351 \times 0.5} = 98.26$. At node I it is worth $100e^{-0.0259 \times 0.5} = 98.71$. At node B it is worth $(0.118 \times 95.70 + 0.654 \times 96.81 + 0.228 \times 97.64)e^{-0.0564 \times 0.5} = 94.17$
Similarly at nodes C and D it is worth 95.60 and 96.68. The value at node A is therefore

\[(0.167 \times 94.17 + 0.666 \times 95.60 + 0.167 \times 96.68)e^{-0.0343 \times 0.5} = 93.92\]

The 18-month zero rate is \(0.08 - 0.05e^{-0.18 \times 1.5} = 0.0418\). This gives the price of the 18-month zero-coupon bond as \(100e^{-0.0418 \times 1.5} = 93.92\) showing that the tree agrees with the initial term structure.

**Problem 30.19.**

What does the calibration of a one-factor term structure model involve?

The calibration of a one-factor interest rate model involves determining its volatility parameters so that it matches the market prices of actively traded interest rate options as closely as possible.

**Problem 30.20.**

Use the DerivaGem software to value \(1 \times 4, 2 \times 3, 3 \times 2,\) and \(4 \times 1\) European swap options to receive fixed and pay floating. Assume that the one, two, three, four, and five year interest rates are 6%, 5.5%, 6%, 6.5%, and 7%, respectively. The payment frequency on the swap is semiannual and the fixed rate is 6% per annum with semiannual compounding. Use the Hull–White model with \(\alpha = 3\%\) and \(\sigma = 1\%\). Calculate the volatility implied by Black’s model for each option.

The option prices are 0.1302, 0.0814, 0.0580, and 0.0274. The implied Black volatilities are 14.28%, 13.64%, 13.24%, and 12.81%

**Problem 30.21.**

Prove equations (30.25), (30.26), and (30.27).

From equation (30.15)

\[P(t, t + \Delta t) = A(t, t + \Delta t)e^{-r(t)B(t, t + \Delta t)}\]

Also

\[P(t, t + \Delta t) = e^{-R(t)\Delta t}\]

so that

\[e^{-R(t)\Delta t} = A(t, t + \Delta t)e^{-r(t)B(t, t + \Delta t)}\]

or

\[e^{-r(t)B(t, T)} = \frac{e^{-R(t)B(t, T)\Delta t}}{A(t, t + \Delta t)B(t, T)/B(t, t + \Delta t)}\]

Hence equation (30.25) is true with

\[\hat{B}(t, T) = \frac{B(t, T)\Delta t}{B(t, t + \Delta t)}\]
and
\[ \dot{A}(t, T) = \frac{A(t, T)}{A(t, t + \Delta t)B(t, T)/B(t, t + \Delta t)} \]
or
\[ \ln \dot{A}(t, T) = \ln A(t, T) - \frac{B(t, T)}{B(t, t + \Delta t)} \ln A(t, t + \Delta t) \]

**ASSIGNMENT QUESTIONS**

**Problem 30.22.**

Construct a trinomial tree for the Ho and Lee model where \( \sigma = 0.02 \). Suppose that the initial zero-coupon interest rate for maturities of 0.5, 1.0, and 1.5 years are 7.5%, 8%, and 8.5%. Use two time steps, each six months long. Calculate the value of a zero-coupon bond with a face value of $100 and a remaining life of six months at the ends of the final nodes of the tree. Use the tree to value a one-year European put option with a strike price of 95 on the bond. Compare the price given by your tree with the analytic price given by DerivaGem.

The tree is shown in Figure M30.1. The probability on each upper branch is 1/6; the probability on each middle branch is 2/3; the probability on each lower branch is 1/6. The six month bond prices nodes E, F, G, H, I are \( 100e^{-0.1442\times0.5} \), \( 100e^{-0.1197\times0.5} \), \( 100e^{-0.0952\times0.5} \), \( 100e^{-0.0707\times0.5} \), and \( 100e^{-0.0462\times0.5} \), respectively. These are 93.04, 94.19, 95.35, 96.53, and 97.71. The payoffs from the option at nodes E, F, G, H, and I are therefore 1.96, 0.81, 0, 0, and 0. The value at node B is \( (0.1667 \times 1.96 + 0.6667 \times 0.81)e^{-0.1095\times0.5} = 0.8380 \). The value at node C is \( 0.1667 \times 0.81 \times e^{-0.0851\times0.5} = 0.1294 \). The value at node D is zero. The value at node A is

\[ (0.1667 \times 0.8380 + 0.6667 \times 0.1294)e^{-0.0750\times0.5} = 0.217 \]

The answer given by DerivaGem using the analytic approach is 0.213.

**Problem 30.23.**

A trader wishes to compute the price of a one-year American call option on a five-year bond with a face value of 100. The bond pays a coupon of 6% semiannually and the (quoted) strike price of the option is $100. The continuously compounded zero rates for maturities of six months, one year, two years, three years, four years, and five years are 4.5%, 5%, 5.5%, 5.8%, 6.1%, and 6.3%. The best fit reversion rate for either the normal or the lognormal model has been estimated as 5%.

A one year European call option with a (quoted) strike price of 100 on the bond is actively traded. Its market price is $0.50. The trader decides to use this option for calibration. Use the DerivaGem software with ten time steps to answer the following questions.
(a) Assuming a normal model, imply the σ parameter from the price of the European option.
(b) Use the σ parameter to calculate the price of the option when it is American.
(c) Repeat (a) and (b) for the lognormal model. Show that the model used does not significantly affect the price obtained providing it is calibrated to the known European price.
(d) Display the tree for the normal model and calculate the probability of a negative interest rate occurring.
(e) Display the tree for the lognormal model and verify that the option price is correctly calculated at the node where, with the notation of Section 30.7, i = 9 and j = −1.

(a) The implied value of σ is 1.12%.
(b) The value of the American option is 0.593.
(c) The implied value of σ is 18.49% and the value of the American option is 0.593. The two models give the same answer providing they are both calibrated to the same European price.
(d) We get a negative interest rate if there are 10 down moves. The probability of this is $0.16667 \times 0.16418 \times 0.16172 \times 0.15928 \times 0.15687 \times 0.15448 \times 0.15218 \times 0.14978 \times 0.14747 \times 0.14518 = 8.3 \times 10^{-9}$
(e) The calculation is

$$0.164179 \times 1.705358 \times e^{-0.052571 \times 0.1} = 0.278516$$

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Problem 30.24.

Use the DerivaGem software to value $1 \times 4$, $2 \times 3$, $3 \times 2$, and $4 \times 1$ European swap options to receive floating and pay fixed. Assume that the one, two, three, four, and five year interest rates are 3%, 3.5%, 3.8%, 4.0%, and 4.1%, respectively. The payment frequency on the swap is semiannual and the fixed rate is 4% per annum with semiannual compounding. Use the lognormal model with $a = 5\%$, $\sigma = 15\%$, and 50 time steps. Calculate the volatility implied by Black's model for each option.

The values of the four European swap options are 1.72, 1.73, 1.30, and 0.65, respectively. The implied Black volatilities are 13.37%, 13.34%, 13.43%, and 13.42%, respectively.

Problem 30.25.

Verify that the DerivaGem software gives Figure 30.11 for the example considered. Use the software to calculate the price of the American bond option for the lognormal and normal models when the strike price is 95, 100, and 105. In the case of the normal model, assume that $a = 5\%$ and $\sigma = 1\%$. Discuss the results in the context of the heaviness of the tails arguments of Chapter 18.

With 100 time steps the lognormal model gives prices of 5.569, 2.433, and 0.699 for strike prices of 95, 100, and 105. With 100 time steps the normal model gives prices of 5.493, 2.511, and 0.890 for the three strike prices respectively. The normal model gives a heavier left tail and a less heavy right tail than the lognormal model for interest rates. This translates into a less heavy left tail and a heavier right tail for bond prices. The arguments in Chapter 18 show that we expect the normal model to give higher option prices for high strike prices and lower option prices for low strike. This is indeed what we find.


Modify Sample Application G in the DerivaGem Application Builder software to test the convergence of the price of the trinomial tree when it is used to price a two-year call option on a five-year bond with a face value of 100. Suppose that the strike price (quoted) is 100, the coupon rate is 7% with coupons being paid twice a year. Assume that the zero curve is as in Table 30.2. Compare results for the following cases:

a. Option is European; normal model with $\sigma = 0.01$ and $a = 0.05$.

b. Option is European; lognormal model with $\sigma = 0.15$ and $a = 0.05$.

c. Option is American; normal model with $\sigma = 0.01$ and $a = 0.05$.

d. Option is American; lognormal model with $\sigma = 0.15$ and $a = 0.05$.

The results are shown in Figure M30.2.
Figure M30.2 Tree for Problem 30.26
CHAPTER 31
Interest Rate Derivatives: HJM and LMM

Notes for the Instructor

This chapter deals with the Heath, Jarrow, and Morton model and Libor Market Model model. (The latter is sometimes also referred to as the Brace, Gatarek, and Musiela or BGM model). This chapter is more mathematically challenging than most other chapters in the book. Instructors who are not teaching advanced students will probably choose to skip it. The chapter deals with multifactor no-arbitrage term structure models. A final section gives some information about the mortgage-backed security market in the United States and option-adjusted spreads. (This section can be taught without the rest of the chapter being covered.)

In addition doing the assignments in the text, the most advanced students can be asked to implement the HJM or LIBOR market model and use it to value some of the mortgage-backed securities described in Section 31.3.

QUESTIONS AND PROBLEMS

Problem 31.1.

Explain the difference between a Markov and a non-Markov model of the short rate.

In a Markov model the expected change and volatility of the short rate at time $t$ depend only on the value of the short rate at time $t$. In a non-Markov model they depend on the history of the short rate prior to time $t$.

Problem 31.2.

Prove the relationship between the drift and volatility of the forward rate for the multifactor version of HJM in equation (31.6).

Equation (31.1) becomes

$$dP(t, T) = r(t)P(t, T)dt + \sum_k v_k(t, T, \Omega_t)P(t, T)dz_k(t)$$

so that

$$d\ln[P(t, T_1)] = \left[r(t) - \sum_k v_k(t, T_1, \Omega_t)\right] dt + \sum_k v_k(t, T_1, \Omega_t) dz_k(t)$$
\[ d\ln[P(t, T_2)] = \left[ r(t) - \sum_k \frac{v_k(t, T_2, \Omega_t)^2}{2} \right] dt + v_k(t, T_2, \Omega_t) dz_k(t) \]

From equation (31.2)

\[ df(t, T_1, T_2) = \sum_k \left[ \frac{v_k(t, T_2, \Omega_t)^2 - v_k(t, T_1, \Omega_t)^2}{2(T_2 - T_1)} \right] dt + \sum_k \frac{v_k(t, T_1, \Omega_t) - v_k(t, T_2, \Omega_t)}{T_2 - T_1} dz_k(t) \]

Putting \( T_1 = T \) and \( T_2 = T + \Delta t \) and taking limits as \( \Delta t \) tends to zero this becomes

\[ dF(t, T) = \sum_k \left[ v_k(t, T, \Omega_t) \frac{\partial v_k(t, T, \Omega_t)}{\partial T} \right] dt - \sum_k \left[ \frac{\partial v_k(t, T, \Omega_t)}{\partial T} \right] dz_k(t) \]

Using \( v_k(t, t, \Omega_t) = 0 \)

\[ v_k(t, T, \Omega_t) = \int_t^T \frac{\partial v_k(t, \tau, \Omega_t)}{\partial \tau} d\tau \]

The result in equation (31.6) follows by substituting

\[ s_k(t, T, \Omega_t) = \frac{\partial v_k(t, T, \Omega_t)}{\partial T} \]

**Problem 31.3.**

"When the forward rate volatility \( s(t, T) \) in HJM is constant, the Ho–Lee model results." Verify that this is true by showing that HJM gives a process for bond prices that is consistent with the Ho–Lee model in Chapter 30.

Using the notation in Section 31.1, when \( s \) is constant,

\[ v_T(t, T) = s \quad v_{TT}(t, T) = 0 \]

Integrating \( v_T(t, T) \)

\[ v(t, T) = sT + \alpha(t) \]

for some function \( \alpha \). Using the fact that \( v(T, T) = 0 \), we must have

\[ v(t, T) = s(T - t) \]

Using the notation from Chapter 30, in Ho–Lee \( P(t, T) = A(t, T)e^{-r(T-t)} \). The standard deviation of the short rate is constant. It follows from Itô’s lemma that the standard deviation of the bond price is a constant times the bond price times \( T - t \). The volatility of the bond price is therefore a constant times \( T - t \). This shows that Ho–Lee is consistent with a constant \( s \).
Problem 3.4.

"When the forward rate volatility, \( s(t, T) \) in HJM is \( \sigma e^{-\alpha(T-t)} \) the Hull-White model results." Verify that this is true by showing that HJM gives a process for bond prices that is consistent with the Hull-White model in Chapter 30.

Using the notation in Section 3.1, when \( v_T(t, T) = s(t, T) = \sigma e^{-\alpha(T-t)} \)
\[
v_{TT}(t, T) = -\alpha \sigma e^{-\alpha(T-t)}
\]
Integrating \( v_T(t, T) \)
\[
v(t, T) = -\frac{1}{\alpha} \sigma e^{-\alpha(T-t)} + \alpha(t)
\]
for some function \( \alpha \). Using the fact that \( v(T, T) = 0 \), we must have
\[
v(t, T) = \frac{\sigma}{\alpha} [1 - e^{-\alpha(T-t)}] = \sigma B(t, T)
\]
Using the notation from Chapter 30, in Hull-White \( P(t, T) = A(t, T)e^{-rB(t, T)} \). The standard deviation of the short rate is constant, \( \sigma \). It follows from Itô's lemma that the standard deviation of the bond price is \( \sigma P(t, T)B(t, T) \). The volatility of the bond price is therefore \( \sigma B(t, T) \). This shows that Hull-White is consistent with \( s(t, T) = \sigma e^{-\alpha(T-t)} \).

Problem 3.5.

What is the advantage of LMM over HJM?

LMM is a similar model to HJM. It has the advantage over HJM that it involves forward rates that are readily observable. HJM involves instantaneous forward rates.

Problem 3.6.

Provide an intuitive explanation of why a ratchet cap increases in value as the number of factors increase.

A ratchet cap tends to provide relatively low payoffs if a high (low) interest rate at one reset date is followed by a high (low) interest rate at the next reset date. High payoffs occur when a low interest rate is followed by a high interest rate. As the number of factors increase, the correlation between successive forward rates declines and there is a greater chance that a low interest rate will be followed by a high interest rate.

Problem 3.7.

Show that equation (31.10) reduces to (31.4) as the \( \delta_i \) tend to zero.

Equation (31.10) can be written
\[
dF_k(t) = \zeta_k(t)F_k(t) \sum_{i=m(t)}^{k} \frac{\delta_i F_i(t)\zeta_i(t)}{1 + \delta_i F_i(t)} dt + \zeta_k(t)F_k(t) dz
\]
As $\delta_i$ tends to zero, $\zeta_i(t)F_i(t)$ becomes the standard deviation of the instantaneous $t_i$-maturity forward rate at time $t$. Using the notation of Section 31.1 this is $s(t, t_i, \Omega_t)$. As $\delta_i$ tends to zero

$$\sum_{i=m(t)}^{k} \frac{\delta_i F_i(t) \zeta_i(t)}{1 + \delta_i F_i(t)}$$

tends to

$$\int_{\tau=t}^{t_k} s(t, \tau, \Omega_t) d\tau$$

Equation (31.10) therefore becomes

$$dF_k(t) = \left[ s(t, t_k, \Omega_t) \int_{\tau=t}^{t_k} s(t, \tau, \Omega_t) d\tau \right] dt + s(t, t_k, \Omega_t) dz$$

This is the HJM result.

**Problem 31.8.**

*Explain why a sticky cap is more expensive than a similar ratchet cap.*

In a ratchet cap, the cap rate equals the previous reset rate, $R$, plus a spread. In the notation of the text it is $R_j + s$. In a sticky cap the cap rate equal the previous capped rate plus a spread. In the notation of the text it is $\min(R_j, K_j) + s$. The cap rate in a ratchet cap is always at least a great as that in a sticky cap. Since the value of a cap is a decreasing function of the cap rate, it follows that a sticky cap is more expensive.

**Problem 31.9.**

*Explain why IOs and POs have opposite sensitivities to the rate of prepayments.*

When prepayments increase, the principal is received sooner. This increases the value of a PO. When prepayments increase, less interest is received. This decreases the value of an IO.

**Problem 31.10.**

*An option adjusted spread is analogous to the yield on a bond.* Explain this statement.

A bond yield is the discount rate that causes the bond’s price to equal the market price. The same discount rate is used for all maturities. An OAS is the parallel shift to the Treasury zero curve that causes the price of an instrument such as a mortgage-backed security to equal its market price.

**Problem 31.11.**

*Prove equation (31.15).*

When there are $p$ factors equation (31.7) becomes

$$dF_k(t) = \sum_{q=1}^{p} \zeta_{k,q}(t) F_k(t) d\sigma_q$$
Equation (31.8) becomes
\[
dF_k(t) = \sum_{q=1}^{p} \zeta_{k,q}(t)[v_{m(t),q} - v_{k+1,q}]F_k(t)dt + \sum_{q=1}^{p} \zeta_{k,q}(t)(F_k(t)dz_q
\]

Equation coefficients of \(dz_q\) in
\[
\ln P(t, t_i) - \ln P(t, t_i+1) = \ln[1 + \delta_i F_i(t)]
\]

Equation (31.9) therefore becomes
\[
\psi_{i,q}(t) - \psi_{i+1,q}(t) = \frac{\delta_i F_i(t)\zeta_{i,q}}{1 + \delta_i F_i(t)}
\]

Equation (31.15) follows.

**Problem 31.12.**

Prove the formula for the variance, \(V(T)\), of the swap rate in equation (31.17).

From the equations in the text
\[
s(t) = \frac{P(t, T_0) - P(t, T_N)}{\sum_{i=0}^{N-1} \tau_i P(t, T_{i+1})}
\]

and
\[
P(t, T_i) = \frac{1}{P(t, T_0)} \sum_{j=0}^{i-1} \frac{1}{1 + \tau_j G_j(t)}
\]

so that
\[
s(t) = \frac{1 - \prod_{j=0}^{N-1} \frac{1}{1 + \tau_j G_j(t)}}{\sum_{i=0}^{N-1} \tau_i \prod_{j=0}^{i} \frac{1}{1 + \tau_j G_j(t)}}
\]

(We employ the convention that empty sums equal zero and empty products equal one.) Equivalently
\[
s(t) = \frac{\prod_{j=0}^{N-1} [1 + \tau_j G_j(t)] - 1}{\sum_{i=0}^{N-1} \tau_i \prod_{j=i+1}^{N-1} [1 + \tau_j G_j(t)]}
\]
or
\[
\ln s(t) = \ln \left\{ \prod_{j=0}^{N-1} [1 + \tau_j G_j(t)] - 1 \right\} - \ln \left\{ \sum_{i=0}^{N-1} \tau_i \prod_{j=i+1}^{N-1} [1 + \tau_j G_j(t)] \right\}
\]

so that
\[
\frac{1}{s(t)} \frac{\partial s(t)}{\partial G_k(t)} = \frac{\tau_k \gamma_k(t)}{1 + \tau_k G_k(t)}
\]
where
\[
\gamma_k(t) = \frac{\prod_{j=0}^{N-1} [1 + \tau_j G_j(t)]}{\prod_{j=0}^{N-1} [1 + \tau_j G_j(t)]} - \frac{\sum_{i=0}^{k-1} \tau_i \prod_{j=i+1}^{N-1} [1 + \tau_j G_j(t)]}{\sum_{i=0}^{N-1} \tau_i \prod_{j=i+1}^{N-1} [1 + \tau_j G_j(t)]}
\]

From Ito’s lemma the \( q \)th component of the volatility of \( s(t) \) is
\[
\sum_{k=0}^{N-1} \frac{1}{s(t) \frac{\partial s(t)}{\partial G_k(t)}} \beta_{k,q}(t) G_k(t) \gamma_k(t)
\]
or
\[
\sum_{k=0}^{N-1} \frac{\tau_k \beta_{k,q}(t) G_k(t) \gamma_k(t)}{1 + \tau_k G_k(t)}
\]
The variance rate of \( s(t) \) is therefore
\[
V(t) = \sum_{q=1}^{p} \left[ \sum_{k=0}^{N-1} \frac{\tau_k \beta_{k,q}(t) G_k(t) \gamma_k(t)}{1 + \tau_k G_k(t)} \right]^2
\]

**Problem 31.13.**

Prove equation (31.19).

\[
1 + \tau_j G_j(t) = \prod_{m=1}^{M} [1 + \tau_{j,m} G_{j,m}(t)]
\]
so that
\[
\ln[1 + \tau_j G_j(t)] = \sum_{m=1}^{M} \ln[1 + \tau_{j,m} G_{j,m}(t)]
\]
Equating coefficients of \( dz_q \)
\[
\frac{\tau_j \beta_{j,q}(t) G_j(t)}{1 + \tau_j G_j(t)} = \sum_{m=1}^{M} \tau_{j,m} \beta_{j,m,q}(t) G_{j,m}(t)
\]
If we assume that \( G_{j,m}(t) = G_{j,m}(0) \) for the purposes of calculating the swap volatility we see from equation (31.17) that the volatility becomes
\[
\sqrt{\frac{1}{T_0} \int_{t=0}^{T_0} \sum_{q=1}^{p} \left[ \sum_{k=n}^{N-1} \sum_{m=1}^{M} \tau_{k,m} \beta_{k,m,q}(t) G_{k,m}(0) \gamma_k(0) \frac{1}{1 + \tau_{k,m} G_{k,m}(0)} \right]^2 dt}
\]
This is equation (31.19).
ASSIGNMENT QUESTIONS


In an annual-pay cap the Black volatilities for caplets with maturities one, two, three, and five years are 18%, 20%, 22%, and 20%, respectively. Estimate the volatility of a one-year forward rate in the LIBOR Market Model when the time to maturity is (a) zero to one year, (b) one to two years, (c) two to three years, and (d) three to five years. Assume that the zero curve is flat at 5% per annum (annually compounded). Use DerivaGem to estimate flat volatilities for two-, three-, four-, five-, and six-year caps.

The cumulative variances for one, two, three, and five years are $0.18^2 \times 1 = 0.0324$, $0.2^2 \times 2 = 0.08$, $0.22^2 \times 3 = 0.1452$, and $0.2^2 \times 5 = 0.2$, respectively. If the required forward rate volatilities are $\Lambda_1$, $\Lambda_2$, $\Lambda_3$, and $\Lambda_4$, we must have

\[
\begin{align*}
\Lambda_1^2 &= 0.0324 \\
\Lambda_2^2 \times 1 &= 0.08 - 0.0324 \\
\Lambda_3^2 \times 1 &= 0.1452 - 0.08 \\
\Lambda_4^2 \times 2 &= 0.2 - 0.1452
\end{align*}
\]

It follows that $\Lambda_1 = 0.18$, $\Lambda_2 = 0.218$, $\Lambda_3 = 0.255$, and $\Lambda_4 = 0.166$

For the last part of the question we first interpolate to obtain the spot volatility for the four-year caplet as 21%. The yield curve is flat at 4.879% with continuous compounding. We use DerivaGem to calculate the prices of caplets with a strike price of 5% where the underlying option matures in one, two, three, four, and five years. The results are 0.3252, 0.4857, 0.6216, 0.6516, and 0.6602, respectively. This means that the prices two-, three-, four-, five-, and six-year caps are 0.3252, 0.8109, 1.4325, 2.0841, and 2.7443. We use DerivaGem again to imply flat volatilities from these prices. The flat volatilities for two-, three-, four-, five-, and six-year caps are 18%, 19.14%, 20.28%, 20.49%, and 20.37%, respectively.

Problem 31.15.

In the flexi cap considered in Section 31.2 the holder is obligated to exercise the first $N$ in-the-money caplets. After that no further caplets can be exercised. (In the example, $N = 5$.) Two other ways that flexi caps are sometimes defined are:

a. The holder can choose whether any caplet is exercised, but there is a limit of $N$ on the total number of caplets that can be exercised.

b. Once the holder chooses to exercise a caplet all subsequent in-the-money caplets must be exercised up to a maximum of $N$.

Discuss the problems in valuing these types of flexi caps. Of the three types of flexi caps, which would you expect to be most expensive? Which would you expect to be least expensive?

The two types of flexi caps mentioned are more difficult to value than the flexi cap considered in Section 31.2. There are two reasons for this.
(i) They are American-style. (The holder gets to choose whether a caplet is exercised.)
This makes the use of Monte Carlo simulation difficult.

(ii) They are path dependent. In (a) the decision on whether to exercise a caplet is liable
to depend on the number of caplets exercised so far. In (b) the exercise of a caplet is
liable to depend on a decision taken some time earlier.

In practice, flexi caps are sometimes valued using a one-factor model of the short rate
in conjunction with the techniques described in Section 26.5 for handling path-dependent
derivatives.

The flexi cap in (b) is worth more than the flexi cap considered in Section 31.2. This
is because the holder of the flexi cap in (b) has all the options of the holder of the flexi
cap in the text and more. Similarly the flexi cap in (a) is worth more than the flexi cap
in (b). This is because the holder of the flexi cap in (a) has all the options of the holder
of the flexi cap in (b) and more. We therefore expect the flexi cap in (a) to be the most
expensive and the flexi cap considered in section 31.2 to be the least expensive.
CHAPTER 32
Swaps Revisited

Notes for the Instructor

This chapter describes a number of nonstandard swap products. These include compounding swaps, currency swaps, LIBOR-in-arrears swaps, CMS and CMT swaps, differential (diff) swaps, equity swaps, accrual swaps, cancelable swaps, index amortizing swaps, commodity swaps, and volatility swaps. One of the aims of the chapter is to distinguish between swaps where the "assume forward rates are realized" rule can be used from swaps where it cannot be used. LIBOR-in-arrears swaps, CMS and CMT swaps, and diff swaps are examples of swaps where the rule cannot be used. As shown in the chapter, the valuation of these deals depend on the convexity, timing, and quanto adjustments explained in Chapters 29.

If Chapter 29 has not been covered it is still possible to teach this chapter by skipping the technical valuation issues. Problem 32.10 and 32.12 can then be used as assignment questions. If the more technical material has been covered Problems 32.9 and 32.11 can be assigned.

QUESTIONS AND PROBLEMS

Problem 32.1.

Calculate all the fixed cash flows and their exact timing for the swap in Business Snapshot 32.1. Assume that the day count conventions are applied using target payment dates rather than actual payment dates.


\[
\frac{181}{365} \times 0.06 \times 100,000,000 = $2,975,342
\]

Similarly subsequent fixed cash flows are: $3,024,658, $2,991,781 $3,041,096, $2,991,781, $2,991,781, $2,991,781, $3,008,219, $2,975,342, and $3,024,658.
Problem 32.2.
Suppose that a swap specifies that a fixed rate is exchanged for twice the LIBOR rate. Can the swap be valued using the “assume forward rates are realized” rule?

Yes. The swap is the same as one on twice the principal where half the fixed rate is exchanged for the LIBOR rate.

Problem 32.3.
What is the value of a two-year fixed-for-floating compound swap where the principal is $100 million and payments are made semiannually? Fixed interest is received and floating is paid. The fixed rate is 8% and it is compounded at 8.3% (both semiannually compounded). The floating rate is LIBOR plus 10 basis points and it is compounded at LIBOR plus 20 basis points. The LIBOR zero curve is flat at 8% with semiannual compounding.

The final fixed payment is in millions of dollars:
\[
[(4 \times 1.0415 + 4) \times 1.0415 + 4] \times 1.0415 + 4 = 17.0238
\]
The final floating payment assuming forward rates are realized is
\[
[(4.05 \times 1.041 + 4.05) \times 1.041 + 4.05] \times 1.041 + 4.05 = 17.2238
\]
The value of the swap is therefore \(-0.2000/(1.04^4) = -0.1710\) or \(-$171,000\).

Problem 32.4.
What is the value of a five-year swap where LIBOR is paid in the usual way and in return LIBOR compounded at LIBOR is received on the other side? The principal on both sides is $100 million. Payment dates on the pay side and compounding dates on the receive side are every six months and the yield curve is flat at 5% with semiannual compounding.

The value is zero. The receive side is the same as the pay side with the cash flows compounded forward at LIBOR. Compounding cash flows forward at LIBOR does not change their value.

Problem 32.5.
Explain carefully why a bank might choose to discount cash flows on a currency swap at a rate slightly different from LIBOR.

In theory, a new floating-for-floating swap should involve exchanging LIBOR in one currency for LIBOR in another currency (with no spreads added). In practice, macroeconomic effects give rise to spreads. Financial institutions often adjust the discount rates they use to allow for this. Suppose that USD LIBOR is always exchanged Swiss franc LIBOR plus 15 basis points. Financial institutions would discount USD cash flows at USD LIBOR and Swiss franc cash flows at LIBOR plus 15 basis points. This would ensure that the floating-for-floating swap is valued consistently with the market.

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Problem 32.6.
Calculate the total convexity/timing adjustment in Example 32.3 of Section 32.4 if all cap volatilities are 18% instead of 20% and volatilities for all options on five-year swaps are 13% instead of 15%. What should the five year swap rate in three years time be assumed for the purpose of valuing the swap? What is the value of the swap?

In this case \( y_i = 0.05, \sigma_{y,i} = 0.13, \tau_i = 0.5, F_i = 0.05, \sigma_{F,i} = 0.18 \), and \( \rho_i = 0.7 \) for all \( i \). It is still true that \( G_i(y_i) = -437.603 \) and \( G_i''(y_i) = 2261.23 \). Equation (32.2) gives the total convexity/timing adjustment as 0.0000892\( t_i \) or 0.892 basis points per year until the swap rate is observed. The swap rate in three years should be assumed to be 5.0268%. The value of the swap is $119,069.

Problem 32.7.
Explain why a plain vanilla interest rate swap and the compounding swap in Section 32.2 can be valued using the “assume forward rates are realized” rule, but a LIBOR-in-arrears swap in Section 32.4 cannot.

In a plain vanilla swap we can enter into a series of FRAs to exchange the floating cash flows for their values if the “assume forward rates are realized rule” is used. In the case of a compounding swap Section 32.2 shows that we are able to enter into a series of FRAs that exchange the final floating rate cash flow for its value when the “assume forward rates are realized rule” is used. There is no way of entering into FRAs so that the floating-rate cash flows in a LIBOR-in-arrears swap are exchanged for their values when the “assume forward rates are realized rule” is used.

Problem 32.8.
In the accrual swap discussed in the text, the fixed side accrues only when the floating reference rate lies below a certain level. Discuss how the analysis can be extended to cope with a situation where the fixed side accrues only when the floating reference rate is above one level and below another.

Suppose that the fixed rate accrues only when the floating reference rate is below \( R_X \) and above \( R_Y \) where \( R_Y < R_X \). In this case the swap is a regular swap plus two series of binary options, one for each day of the life of the swap. Using the notation in the text, the risk-neutral probability that LIBOR will be above \( R_X \) on day \( i \) is \( N(d_2) \) where

\[
d_2 = \frac{\ln(F_i/R_X) - \sigma_i^2 t_i^2/2}{\sigma_i \sqrt{t_i}}
\]

The probability that it will be below \( R_Y \) where \( R_Y < R_X \) is \( N(-d'_2) \) where

\[
d'_2 = \frac{\ln(F_i/R_Y) - \sigma_i^2 t_i^2/2}{\sigma_i \sqrt{t_i}}
\]

From the viewpoint of the party paying fixed, the swap is a regular swap plus binary options. The binary options corresponding to day \( i \) have a total value of

\[
\frac{QL}{n_2} P(0,s_i)[N(d_2) + N(-d'_2)]
\]
ASSIGNMENT QUESTIONS

Problem 32.9.

LIBOR zero rates are flat at 5% in the U.S and flat at 10% in Australia (both annually compounded). In a four-year swap Australian LIBOR is received and 9% is paid with both being applied to a USD principal of $10 million. Payments are exchanged annually. The volatility of all one-year forward rates in Australia is estimated to be 25%, the volatility of the forward USD–AUD exchange rate (AUD per USD) is 15% for all maturities, and the correlation between the two is 0.3. What is the value of the swap?

The fixed side consists of four payments of USD 0.9 million. The present value in millions of dollars is

\[
\frac{0.9}{1.05} + \frac{0.9}{1.05^2} + \frac{0.9}{1.05^3} + \frac{0.9}{1.05^4} = 2.85
\]

The forward Australian LIBOR rate is 10% with annual compounding. From Section 29.3 the quanto adjustment to the floating payment at time \(t_i + 1\) is

\[
0.1 \times 0.7 \times 0.25t_i = 0.002625t_i
\]

The value of the floating payments is therefore

\[
\frac{1}{1.05} + \frac{1.02625}{1.05^2} + \frac{1.0525}{1.05^3} + \frac{1.07875}{1.05^4} = 3.28
\]

The value of the swap is 3.28 – 2.85 = 0.43.

Problem 32.10.

Estimate the interest rate paid by P&G on the 5/30 swap in Section 32.7 if a) the CP rate is 6.5% and the Treasury yield curve is flat at 6% and b) the CP rate is 7.5% and the Treasury yield curve is flat at 7% with semiannual compounding.

When the CP rate is 6.5% and Treasury rates are 6% with semiannual compounding, the CMT% is 6% and an Excel spreadsheet can be used to show that the price of a 30-year bond with a 6.25% coupon is about 103.46. The spread is zero and the rate paid by P&G is 5.75%. When the CP rate is 7.5% and Treasury rates are 7% with semiannual compounding, the CMT% is 7% and the price of a 30-year bond with a 6.25% coupon is about 90.65. The spread is therefore

\[
\max[0, (98.5 \times 7/5.78 - 90.65)/100]
\]

or 28.64%. The rate paid by P&G is 35.39%.
Problem 32.11.

Suppose that you are trading a LIBOR-in-arrears swap with an unsophisticated counterparty who does not make convexity adjustments. To take advantage of the situation, should you be paying fixed or receiving fixed? How should you try to structure the swap as far as its life and payment frequencies?

Consider the situation where the yield curve is flat at 10% per annum with annual compounding. All cap volatilities are 18%. Estimate the difference between the way a sophisticated trader and an unsophisticated trader would value a LIBOR-in-arrears swap where payments are made annually and the life of the swap is (a) 5 years, (b) 10 years, and (c) 20 years. Assume a notional principal of $1 million.

You should be paying fixed and receiving floating. The counterparty will value the floating payments less than you because it does not make a convexity adjustment increasing forward rates. The size of the convexity adjustment for a forward rate increases with the forward rate, the forward rate volatility, the time between resets, and the time until the forward rate is observed. We therefore maximize the impact of the convexity adjustment by choosing long swaps involving high interest rate currencies, where the interest rate volatility is high and there is a long time between resets.

The convexity adjustment for the payment at time \( t_i \) is

\[
0.1^2 \times 0.18^2 \times 1 \times t_i \times 1.1
\]

This is \( 0.000295t_i \). For a five year LIBOR-in-arrears swap the value of the convexity adjustment is

\[
1,000,000 \times \sum_{i=1}^{5} \frac{0.000295i}{1.1^i}
\]

or $3137.7. Similarly the value of the convexity adjustments for 10 and 20 year swaps is $8,552.4 and $18,827.5.

Problem 32.12.

Suppose that the LIBOR zero rate is flat at 5% with annual compounding. In a five-year swap, company X pays a fixed rate of 6% and receives LIBOR. The volatility of the two-year swap rate in three years is 20%.

a. What is the value of the swap?
b. Use DerivaGem to calculate the value of the swap if company X has the option to cancel after three years.
c. Use DerivaGem to calculate the value of the swap if the counterparty has the option to cancel after three years.
d. What is the value of the swap if either side can cancel at the end of three years?

(a) Because the LIBOR zero curve is flat at 5% with annual compounding, the five-year swap rate for an annual-pay swap is also 5%. (As explained in Chapter 7 swap rates are par yields.) A swap where 5% is paid and LIBOR is receive would therefore be
worth zero. A swap where 6% is paid and LIBOR is received has the same value as an instrument that pays 1% per year. Its value in millions of dollars is therefore

\[- \frac{1}{1.05} - \frac{1}{1.05^2} - \frac{1}{1.05^3} - \frac{1}{1.05^4} - \frac{1}{1.05^5} = -4.33\]

(b) In this case company X has, in addition to the swap in (a), a European swap option to enter into a two-year swap in three years. The swap gives company X the right to receive 6% and pay LIBOR. We value this in DerivaGem by using the Caps and Swap Options worksheet. We choose Swap Option as the Underlying Type, set the Principal to 100, the Settlement Frequency to Annual, the Swap Rate to 6%, and the Volatility to 20%. The Start (Years) is 3 and the End (Years) is 5. The Pricing Model is Black-European. We choose Rec Fixed and do not check the Imply Volatility or Imply Breakeven Rate boxes. All zero rates are 4.879% with continuous compounding. We therefore need only enter 4.879% for one maturity. The value of the swap option is given as 2.18. The value of the swap with the cancelation option is therefore

\[-4.33 + 2.18 = -2.15\]

(c) In this case company X has, in addition to the swap in (a), granted an option to the counterparty. The option gives the counterparty the right to pay 6% and receive LIBOR on a two-year swap in three years. We can value this in DerivaGem using the same inputs as in (b) but with the Pay Fixed instead of the Rec Fixed being chosen. The value of the swap option is 0.57. The value of the swap to company X is

\[-4.33 - 0.57 = -4.90\]

(d) In this case company X is long the Rec Fixed option and short the Pay Fixed option. The value of the swap is therefore

\[-4.33 + 2.18 - 0.57 = -2.72\]

It is certain that one of the two sides will exercise its option to cancel in three years. The swap is therefore to all intents and purposes a three-year swap with no embedded options. Its value can also be calculated as

\[- \frac{1}{1.05} - \frac{1}{1.05^2} - \frac{1}{1.05^3} = -2.72\]
CHAPTER 33
Real Options

Notes for the Instructor

The real options approach to capital investment appraisal has become so popular in recent years that it is clearly appropriate to include a chapter on it in a derivatives text. It is potentially much easier to value embedded options, such as expansion and abandonment options, using the real options approach than using traditional approaches. (Some people argue that real options including the whole of options pricing should be a central part of any course in corporate finance!)

I use two examples to illustrate the real options approach. The first is the Schwartz and Moon study aimed at valuing Amazon.com. The second is an oil exploration example that I developed myself. I find that students relate well to both examples. The oil exploration example also serves as a vehicle to illustrate how a trinomial tree can be built for a commodity price.

Both Problem 33.8 and 33.9 work well as assignment questions.

QUESTIONS AND PROBLEMS

Problem 33.1.

Explain the difference between the net present value approach and the risk-neutral valuation approach for valuing a new capital investment opportunity. What are the advantages of the risk-neutral valuation approach for valuing real options?

In the net present value approach, cash flows are estimated in the real world and discounted at a risk-adjusted discount rate. In the risk-neutral valuation approach, cash flows are estimated in the risk-neutral world and discounted at the risk-free interest rate. The risk-neutral valuation approach is arguably more appropriate for valuing real options because it is very difficult to determine the appropriate risk-adjusted discount rate when options are valued.

Problem 33.2.

The market price of risk for copper is 0.5, the volatility of copper prices is 20% per annum, the spot price is 80 cents per pound, and the six-month futures price is 75 cents per pound. What is the expected percentage growth rate in copper prices over the next six months?

In a risk-neutral world the expected price of copper in six months is 75 cents. This corresponds to an expected growth rate of $2 \ln(75/80) = -12.9\%$ per annum. The decrease
in the growth rate when we move from the real world to the risk-neutral world is the volatility of copper times its market price of risk. This is $0.2 \times 0.5 = 0.1$ or 10% per annum. It follows that the expected growth rate of the price of copper in the real world is $-2.9\%$.

**Problem 33.3.**

*Consider a commodity with constant volatility, $\sigma$, and an expected growth rate that is a function solely of time. Show that in the traditional risk-neutral world,*

$$\ln S_T \sim \phi \left[ \ln F(T) - \frac{\sigma^2}{2}T, \sigma^2T \right]$$

*where $S_T$ is the value of the commodity at time $T$, $F(t)$ is the futures price at time zero for a contract maturing at time $t$, and $\phi(m, v)$ is a normal distribution with mean $m$ and variance $v$. 

In this case

$$\frac{dS}{S} = \mu(t) dt + \sigma dz$$

or

$$d\ln S = [\mu(t) - \sigma^2 / 2] dt + \sigma dz$$

so that $\ln S_T$ is normal with mean

$$\ln S_0 + \int_{t=0}^{T} \mu(t) dt - \sigma^2 T / 2$$

and standard deviation $\sigma \sqrt{T}$. Section 33.5 shows that

$$\mu(t) = \frac{\partial}{\partial t} [\ln F(t)]$$

so that

$$\int_{t=0}^{T} \mu(t) dt = \ln F(T) - \ln F(0)$$

Since $F(0) = S_0$ the result follows.

**Problem 33.4.**

*Derive a relationship between the convenience yield of a commodity and its market price of risk.*

We explained the concept of a convenience yield for a commodity in Chapter 5. It is a measure of the benefits realized from ownership of the physical commodity that are not realized by the holders of a futures contract. If $y$ is the convenience yield and $u$ is the storage cost, equation (5.17) shows that the commodity behaves like an investment asset that provides a return equal to $y - u$. In a risk-neutral world its growth is, therefore,

$$r - (y - u) = r - y + u$$
The convenience yield of a commodity can be related to its market price of risk. From Section 33.2, the expected growth of the commodity price in a risk-neutral world is \( m - \lambda s \), where \( m \) is its expected growth in the real world, \( s \) its volatility, and \( \lambda \) is its market price of risk. It follows that

\[
m - \lambda s = r - y + u
\]

or

\[
y = r + u - m + \lambda s
\]

**Problem 33.5.**

The correlation between a company’s gross revenue and the market index is 0.2. The excess return of the market over the risk-free rate is 6% and the volatility of the market index is 18%. What is the market price of risk for the company’s revenue?

In equation (33.2) \( \rho = 0.2, \mu_m - r = 0.06 \), and \( \sigma_m = 0.18 \). It follows that the market price of risk lambda is

\[
\frac{0.2 \times 0.06}{0.18} = 0.067
\]

**Problem 33.6.**

A company can buy an option for the delivery of one million units of a commodity in three years at $25 per unit. The three year futures price is $24. The risk-free interest rate is 5% per annum with continuous compounding and the volatility of the futures price is 20% per annum. How much is the option worth?

The option can be valued using Black’s model. In this case \( F_0 = 24, K = 25, r = 0.05, \sigma = 0.2 \), and \( T = 3 \). The value of a option to purchase one unit at $25 is

\[
e^{-rT}[F_0N(d_1) - KN(d_2)]
\]

where

\[
d_1 = \frac{\ln(F_0/K) + \sigma^2T/2}{\sigma\sqrt{T}}
\]

\[
d_2 = \frac{\ln(F_0/K) - \sigma^2T/2}{\sigma\sqrt{T}}
\]

This is 2.489. The value of the option to purchase one million units is therefore $2,489,000.

**Problem 33.7.**

A driver entering into a car lease agreement can obtain the right to buy the car in four years for $10,000. The current value of the car is $30,000. The value of the car, \( S \), is expected to follow the process

\[
dS = \mu S dt + \sigma S dz
\]

where \( \mu = -0.25, \sigma = 0.15 \), and \( dz \) is a Wiener process. The market price of risk for the car price is estimated to be \(-0.1\). What is the value of the option? Assume that the risk-free rate for all maturities is 6%.
The expected growth rate of the car price in a risk-neutral world is \(-0.25 - (-0.1 \times 0.15) = -0.235\). The expected value of the car in a risk-neutral world in four years, \(\hat{E}(S_T)\), is therefore \(30,000e^{-0.235 \times 4} = \$11,719\). Using the result in the appendix to Chapter 13, the value of the option is

\[ e^{-rT}[\hat{E}(S_T)N(d_1) - KN(d_2)] \]

where

\[ d_1 = \frac{\ln(\hat{E}(S_T)/K) + \sigma^2T/2}{\sigma\sqrt{T}} \]
\[ d_2 = \frac{\ln(\hat{E}(S_T)/K) - \sigma^2T/2}{\sigma\sqrt{T}} \]

\(r = 0.06, \sigma = 0.15, T = 4, \) and \(K = 10,000\). It is \$1,832.

**ASSIGNMENT QUESTIONS**

**Problem 33.8.**

Suppose that the spot price, 6-month futures price, and 12-month futures price for wheat are 250, 260, and 270 cents per bushel, respectively. Suppose that the price of wheat follows the process in equation (33.4) with \(a = 0.05\) and \(\sigma = 0.15\). Construct a two-time-step tree for the price of wheat in a risk-neutral world.

A farmer has a project that involves an expenditure of \$10,000 and a further expenditure of \$90,000 in six months. It will increase wheat that is harvested and sold by 40,000 bushels in one year. What is the value of the project? Suppose that the farmer can abandon the project in six months and avoid paying the \$90,000 cost at that time. What is the value of the abandonment option? Assume a risk-free rate of 5% with continuous compounding.

In this case \(a = 0.05\) and \(\sigma = 0.15\). We first define a variable \(X\) that follows the process

\[ dX = -adt + \sigma dz \]

A tree for \(X\) constructed in the way described in Chapter 28 is shown in Figure M33.1. We now displace nodes so that the tree models \(\ln S\) in a risk-neutral world where \(S\) is the price of wheat. The displacements are chosen so that the initial price of wheat is 250 cents and the expected prices at the ends of the first and second time steps are 260 and 270 cents, respectively. Suppose that the displacement to give \(\ln S\) at the end of the first time step is \(\alpha_1\). Then

\[ 0.1667e^{\alpha_1 + 0.1837} + 0.6666e^{\alpha_1} + 0.1667e^{\alpha_1 - 0.1837} = 260 \]

so that \(\alpha_1 = 5.5551\). The probabilities of nodes E, F, G, H, I being reached are 0.0257, 0.2221, 0.5043, 0.2221, and 0.0257, respectively. Suppose that the displacement to give \(\ln S\) at the end of the second step is \(\alpha_2\). Then

\[ 0.0257e^{\alpha_2 + 0.3674} + 0.2221e^{\alpha_2 + 0.1837} + 0.5043e^{\alpha_2} + 0.2221e^{\alpha_2 - 0.1837} \]
so that $\alpha_2 = 5.5874$. This leads to the tree for the price of wheat shown in Figure M33.2.

Using risk-neutral valuation the value of the project (in thousands of dollars) is

$$-10 - 90e^{-0.05 \times 0.5} + 2.70 \times 40e^{-0.05 \times 1} = 4.94$$

This shows that the project is worth undertaking. Figure M33.3 shows the value of the project on a tree. The project should be abandoned at node D for a saving of 2.41. Figure M33.4 shows that the abandonment option is worth 0.39.

**Problem 33.9.**

*In the example considered in Section 33.6*

a. What is the value of the abandonment option if it costs $3$ million rather than zero?

b. What is the value of the expansion option if it costs $5$ million rather than $2$ million?

Figure M33.5 shows what Figure 33.4 in the text becomes if the abandonment option costs $3$ million. The value of the abandonment option reduces from 1.94 to 1.21. Similarly Figure M33.6 below shows what Figure 33.5 in the text becomes if the expansion option costs $5$ million. The value of the expansion option reduces from 1.06 to 0.40.

![Tree for X in Problem 33.8](Figure M33.1)
Figure M33.2  Tree for price of wheat in Problem 33.8

Figure M33.3  Tree for value of project in Problem 33.8
Figure M33.4  Tree for abandonment option in Problem 33.8

Figure M33.5  Tree for abandonment option in Problem 33.9
Figure M33.6  Tree for expansion option in Problem 33.9
CHAPTER 34
Derivatives Mishaps and What We Can Learn from Them

Notes for the Instructor

This chapter describes the well-publicized derivatives disasters of the last 20 years and discusses the lessons that can be learned from them.

Chapter 32 is a great chapter for the final class of a course. Students love talking about the derivatives disasters of the past, what went wrong, why it went wrong, etc. I find that questions about some of the disasters (particularly LTCM) often arise relatively early in my courses. It is useful to be able to say that we are going to talk about them in some detail towards the end of the semester.

I like to go through the lessons one by one. When a derivatives disaster is relevant to a particular lesson, we talk about it in some detail. It is not a good idea to sound too self-righteous when discussing the disasters. It is a good idea to say things like “Of course it is easy to be wise after the event.” Asking students whether they think the derivatives business has learned its lessons or whether there will be more disasters in the future can generate interesting viewpoints.

I like to try to end on a positive note such as: “The derivatives industry is huge and it is here to stay. These disasters are not representative of what happens in the industry. They constitute a very very small proportion of the total trades. Most trades are entered into by corporations for sensible hedging and risk management purposes. But the disasters are interesting because of what we can learn from them.”
Test Bank Questions

Pages 420 to 446 contain test bank questions for chapters 1 to 21. Some questions are multiple choice; others have numerical answers that can easily be computed using a calculator. In each of the 21 tests, there are a total of ten answers that can be quickly graded as correct or incorrect.

The questions can be used at the beginning of a class to provide a quick test of whether students understand the material covered in the previous class. Alternatively, they can be used to provide instructors with ideas for midterm or final exam questions.

The answers are on pages 447–448.

Microsoft Word files for the tests can be downloaded from the Pearson Instructor Resource Center.